Lecture 19: 7.2 Derivation of the flow integral. Let S be a closed surface and for each point (x, y, z) on the surface let $f(x, y, z)$ be the rate of flow of fluid out through the surface per unit surface area and unit time, i.e. the flow out of a small area $\triangle S$ during a small time $\triangle t$ is approximately $f\,\triangle S \triangle t$. The total flow of fluid out from the region enclosed by the surface per unit time is the **surface integral**

$$\iint_S f\, dS = \lim_{\triangle S_{ij} \to 0} \sum_{i,j} f(x_{ij}, y_{ij}, z_{ij}) \triangle S_{ij}$$

where the sum is over a partition of S into smaller surface areas $\triangle S_{ij}$, (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$ and we take the limit as the partition becomes finer.

Let us now calculate the rate of flow of fluid f out per unit area and unit time, given the velocity vector field of the fluid $\mathbf{V}$ and the density μ. We define the **flow rate density** by

$$\mathbf{F} = \mu \mathbf{V},$$

If $\triangle S$ is a small area of a piece of a plane with outward unit normal $\mathbf{n}$ then we claim that the flow rate out of $\triangle S$ per unit time is given by

$$\mathbf{F} \cdot \mathbf{n}\, \triangle S$$

In fact, in a small time $\triangle t$, the fluid particles that will reach $\triangle S$ are at most $\mathbf{V} \triangle t$ away, and all particles within reach form a sloped cylinder with $\triangle S$ as its base and height $\mathbf{V} \cdot \mathbf{n}\, \triangle t$. Since the volume is the area of the base times the height the amount of fluid in the cylinder is the density times the volume: $\mu \mathbf{V} {\cdot} \mathbf{n}\, \triangle t \triangle S$. If we divide by $\triangle t$ we get the rate per unit time and if we divide this by $\triangle S$ we get the flow rate out of per unit surface area and unit time

$$f = \mathbf{F} \cdot \mathbf{n}$$

The flow rate of fluid out of the total surface S, or the **flux** of the velocity vector field $\mathbf{F}$ out of the surface S, with outward unit normal $\mathbf{n}$, is the **surface integral**

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

Examples of calculating flow integrals

Ex Let R be the 3-dimensional region $R = \{x^2/4 + y + z^2/4 \le 1,\, y \ge 0\}$. Let S be the surface of R with the normal oriented outwards. Note that S has two parts $S_1 = \{y = 0,\, x^2/4 + z^2/4 \le 1\}$ and $S_2 = \{x^2/4 + y + z^2/4 = 1,\, y \ge 0\}$.

a) Find the area of S.
b) Find the flux of $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + \mathbf{k}$ through S; $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$.

Sol (a) The area of S_1 is 4π.

S_2 can be viewed as a graph $y = g(x,z) = 1 - x^2/4 - z^2/4$ over the disc $D = \{(x,z);\, x^2/4 + z^2/4 \le 1\}$ in the xz-plane. With $G(x,y,z) = y - g(x,z)$ the unit normal is $\mathbf{n} = \dfrac{\nabla G}{|\nabla G|} = \dfrac{-g_x\mathbf{i} - g_z\mathbf{k} + \mathbf{j}}{\sqrt{1 + g_x^2 + g_z^2}} = \dfrac{x\mathbf{i}/2 + z\mathbf{k}/2 + \mathbf{j}}{\sqrt{1 + x^2/4 + z^2/4}}$.
Now $dS = \dfrac{dxdz}{\mathbf{n}\cdot\mathbf{j}} = \sqrt{1 + (x^2+z^2)/4}\,dxdz$.

Introducing polar coordinates in the xz-plane:

$$\iint_{S_2} dS = \iint_D \Big(1+\frac{x^2+z^2}{4}\Big)^{1/2} dxdz = \int_0^2\int_0^{2\pi}\Big(1+\frac{r^2}{4}\Big)^{1/2} d\theta r dr = 2\pi\frac{4}{3}\Big(1+\frac{r^2}{4}\Big)^{3/2}\Big|_0^2 = \frac{8\pi}{3}(2^{3/2}-1)$$

(b) The normal to S_1 is $\mathbf{n} = -\mathbf{j}$ and $\mathbf{F}\cdot\mathbf{n} = y = 0$ so the integral over S_1 vanishes.

Since $\mathbf{F}\cdot\mathbf{n} = (x^2/2 + z/2 - y)/\sqrt{1 + x^2/4 + z^2/4}$ we obtain

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D (x^2/2 + z/2 - y)dxdz = \iint_D \big(x^2/2 + z/2 - (1 - (x^2+z^2)/4)\big)\,dxdz$$

Introducing polar coordinates in the xz-plane; $x = r\cos\theta$, $z = r\sin\theta$, $dxdz = rdrd\theta$;

$$\int_0^2\int_0^{2\pi}\frac{r^2}{4}(2\cos^2\theta+1) + \frac{r}{2}\sin\theta - 1\,d\theta\,r dr = \int_0^2\int_0^{2\pi}\frac{r^2}{4}(\cos 2\theta + 2) - \frac{r}{2}\sin\theta - 1\,d\theta\,r dr = \ldots = 0$$

Alternative solution: Let $\mathbf{X}(x,z) = x\mathbf{i} + g(x,z)\mathbf{j} + z\mathbf{k}$ and

$$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \pm\iint_{x^2+z^2\le 4} \mathbf{F}\cdot(\mathbf{X}_x \times \mathbf{X}_z)\,dxdz = \ldots$$

7.3 Gauss' Theorem or The divergence theorem. states that if D is a volume bounded by a surface S with outward unit normal $\mathbf{n}$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a continuously differentiable vector field in D then

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \operatorname{div}\mathbf{F}\,dV, \qquad \text{where} \quad \operatorname{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

A one dimensional analogue is the First Fundamental Theorem of Calculus:

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

A two dimensional analogue says that if D is a region in the plane with boundary curve C and $\mathbf{n} = (n_1, n_2)$ is the outward unit normal to C, then

$$\int_C F_1 n_1 + F_2 n_2 \; ds = \iint_D \Big(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\Big)\,dA,$$

where ds is the arclength. (This is in fact equivalent to Green's Theorem with $F_1 = Q$ and $F_2 = -P$ and $\mathbf{n} = (dy/ds, -dx/ds)$, where $\mathbf{T} = (dx/ds, dy/ds)$ is the unit tangent vector to the boundary curve, which follows since then $\mathbf{T}\cdot\mathbf{n} = 0$.)

Ex. Find flux of $\mathbf{F} = 2x\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ out of the unit sphere S.
Sol. By the divergence theorem we have with B the unit ball

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_B \operatorname{div} F\,dV \; &= \iiint_B (2 + 2y + 2z)\,dV \\ = \iiint_B 2dV + \iiint_B 2ydV + \iiint_B 2zdV &= 2\operatorname{Vol}(B) + 0 + 0 = 2\frac{4\pi}{3}\end{aligned}$$

since the last two integrals vanishes because the region is symmetric under replacing y by $-y$ (respectively z by $-z$) but the integrand changes sign.

Ex. Find flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ out of the unit sphere.
Sol. By the divergence theorem the flux is equal to the integral of the divergence over the unit ball. Since $\operatorname{div}\mathbf{F} = 0$ it follows that the volume integral vanishes and by the divergence theorem the flux therefore vanishes.

Physical Interpretation of Divergence Let B_ε be a ball of radius ε and let S_ε be its surface. Then

$$\iint_{S_\varepsilon} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{B_\varepsilon} \operatorname{div}\mathbf{F}\,dV.$$

By the **mean value theorem for integrals** the right hand side is equal to the volume of the ball B_ε times $\operatorname{div}\mathbf{F}$ at some point in the ball so

$$\text{Flux of } \mathbf{F} \text{ out through } S_\varepsilon = \operatorname{Vol}(B_\varepsilon)\operatorname{div}\mathbf{F},$$

where $\operatorname{div}\mathbf{F}$ is evaluated at some point in B_ε. Hence

$$\operatorname{div}\mathbf{F} = \lim_{\varepsilon\to 0} \frac{\text{Flux of } \mathbf{F} \text{ out through } S_\varepsilon}{\operatorname{Vol}(B_\varepsilon)}.$$

Proof of the divergence theorem for convex sets.
We say that a domain D is **convex** if for every two points in D the line segment between the two points is also in D, e.g. any sphere or rectangular box is convex.

We will prove the divergence theorem for convex domains D. Since $\mathbf{F} = F_1\mathbf{i} + F_3\mathbf{j} + F_3\mathbf{k}$ the theorem follows from proving the theorem for each of the three vector fields $F_1\mathbf{i}$, $F_2\mathbf{j}$ and $F_3\mathbf{k}$ separately. The theorem for the vector field $F_3\mathbf{k}$ states that

$$\iint_S (F_3\mathbf{k})\cdot\mathbf{n}\,dS = \iiint_D \frac{\partial F_3}{\partial z}\,dV$$

Since D is convex we can write $D = \{(x,y,z);\ f_1(x,y) \le z \le f_2(x,y),\ (x,y) \in D\}$.

Then S consists of two parts $S_1 = \{(x,y,z);\ z = f_1(x,y),\ (x,y) \in D\}$ and $S_2 = \{(x,y,z);\ z = f_2(x,y),\ (x,y) \in D\}$. We have

$$\begin{aligned}\iiint_V \frac{\partial F_3}{\partial z}\,dzdxdy &= \iint_D \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z}\,dz\ dxdy\\ &= \iint_D F_3\big((x,y,f_2(x,y)\big)\,dxdy - \iint_D F_3\big((x,y,f_1(x,y)\big)\,dxdy\end{aligned}$$

We know that $dxdy = \mathbf{k}\cdot\mathbf{n}\,dS$ on S_2 and $dxdy = -\mathbf{k}\cdot\mathbf{n}\,dS$ on S_1 so (5.1.4) follows.

Ex. Gauss law Let $\mathbf{F} = -(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2+y^2+z^2)^{3/2}$. Let S_a be the sphere of radius a centered at the origin. Find the flux of $\mathbf{F}$ out of S_a

Sol. A calculation shows that $\operatorname{div}\mathbf{F} = ... = 0$, when $|x| \ne 0$. Hence by the divergence theorem the flux is $\iint_{S_a} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_{B_a} \operatorname{div}\mathbf{F}\,dV$, where B_a is the ball of radius a centered at 0. From this we deduce that the flux is 0 but this is wrong!

In fact the outward unit normal to $S_a = \{(x,y,z);\ x^2+y^2+z^2 = a^2\}$ is $\mathbf{n} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2+y^2+z^2)^{1/2}$. It therefore follows that

$$\iint_{S_a} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_a} \frac{dS}{x^2+y^2+z^2} = \iint_{S_a} \frac{1}{a^2}\,dS = \frac{1}{a^2}\text{Area}\,(S_a) = \frac{1}{a^2}4\pi a^2 = 4\pi$$

Therefore we conclude that the divergence theorem is not valid in this case. In fact $\mathbf{F}$ has to be continuously differentiable and bounded in $\overline{D}$ for the theorem to hold, but $\mathbf{F}$ is unbounded at the origin. We also remark that the flux of $\mathbf{F}$ out of any region that contains the origin is 4π, and 0 if the region does not contain the origin.

Mean value theorem for integrals If f is continuous in D then

$$\iiint_D f\,dV = f(x_0,y_0,z_0)\,\text{Volume}(D), \quad \text{for some } (x_0,y_0,z_0) \in D.$$

In fact if $\min_D f \le f(x,y,z) \le \max_D f$ for $(x,y,z) \in D$ so the integral is between $\min_D f \iiint_D dV \le \iiint_D f dV \le \max_D f \iiint_D dV$. Since f is continuous there is $(x_0,y_0,z_0) \in D$ so it is equal to $f(x_0,y_0,z_0)\iiint_D dV$, where $\iiint_D dV = \text{Volume}(D)$.