Lecture 18: 7.2 Surface Integrals Suppose we want to find the total volume of water in the oceans of the earth. At each point of the surface the depth of the ocean is given by a function f.
To measure the total volume we divide up the surface S into smaller surface areas $\triangle S_{ij}$, each of which is so small that we can think of it as approximately flat under which the depth of the ocean is approximately constant. The volume of water below $\triangle S_{ij}$ is approximately $f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$, where (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$.

The total volume of water in the ocean is approximately $\sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$. We therefore define the **surface integral** of a function f over the surface S to be

$$\iint_S f\, dS = \lim_{\triangle S_{ij}\to 0} \sum_{i,j} f(x_{ij}, y_{ij}, z_{ij})\triangle S_{ij}$$

where the sum is over a partition of S into smaller surface areas $\triangle S_{ij}$, (x_{ij}, y_{ij}, z_{ij}) is any point in $\triangle S_{ij}$ and we take the limit as the partition becomes finer.

Suppose that S is a parameterized surface: $\mathbf{X}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$, where $(u, v) \in D$. Let $R_{ij} = \{(u, v);\ u_i \le u \le u_i + \triangle u,\ v_j \le v \le v_j + \triangle v\}$ be a small rectangle in the u-v plane and let S_{ij} be the image of R_{ij} under the map $(u, v) \to \mathbf{X}(u, v)$. Then the area of S_{ij} is approximately the area in of the parallelogram in the tangent plane spanned by the vectors $\mathbf{X}_u\triangle u$ and $\mathbf{X}_v\triangle v$:

$$\triangle S_{ij} \sim \big\|\mathbf{X}_u \times \mathbf{X}_v(u_i, v_j)\big\|\triangle u\triangle v$$

Substituting this we get a Riemann sum for a double integral in the u-v plane. We therefore define the surface integral of a function f over a surface S:

$$\iint_S f\, dS = \iint_D f\big(x(u, v), y(u, v), z(u, v)\big)\big\|\mathbf{X}_u \times \mathbf{X}_v(u, v)\big\|\, dudv$$

We can symbolically write

$$dS = \big\|\mathbf{X}_u \times \mathbf{X}_v(u, v)\big\|\, dudv$$

Ex. Find $\iint_S z^2\,dS$, where $S=\{(x,y,z);\, x^2+y^2+z^2=1\}$ is the unit sphere.
Sol. 1 A parametrization is $\mathbf{X}(\phi,\theta)=\sin\phi\cos\theta\,\mathbf{i}+\sin\phi\sin\theta\,\mathbf{j}+\cos\phi\,\mathbf{k}$, $0\le\phi\le\pi$, $0\le\theta<2\pi$, and we showed before that

$$dS=\big\|\mathbf{X}_\phi\times\mathbf{X}_\theta(\phi,\theta)\big\|d\phi\,d\theta=\sin\phi\,d\phi\,d\theta$$

Hence

$$\iint_S z^2\,dS=\int_0^{2\pi}\!\!\int_0^{\pi}\cos^2\phi\,\sin\phi\,d\phi\,d\theta=\int_0^{2\pi}-\frac{\cos^3\phi}{3}\Big|_0^{\pi}\,d\theta=\int_0^{2\pi}\frac{2}{3}\,d\theta=\frac{4\pi}{3}$$

Sol. 2 A parametrization of the northern hemisphere S_+ is $\mathbf{X}(x,y)=x\,\mathbf{i}+y\,\mathbf{j}+f(x,y)\,\mathbf{k}$, where $f(x,y)=\sqrt{1-x^2-y^2}$ and

$$dS=\|\mathbf{X}_x\times\mathbf{X}_y\|\,dxdy=\sqrt{1+f_x^2+f_y^2}\,dxdy=\frac{dxdy}{\sqrt{1-x^2-y^2}}$$

Since the integral of z^2 over the southern and northern hemispheres are the same

$$\begin{aligned}\int_S z^2\,dS&=2\int_{S_+}z^2\,dS=2\iint_{x^2+y^2\le1}(1-x^2-y^2)\frac{dxdy}{\sqrt{1-x^2-y^2}}\\&=2\int_0^{2\pi}\int_0^1(1-r^2)^{1/2}r\,d\,d\theta=2\int_0^{2\pi}\frac{1}{3}(1-r^2)^{3/2}\Big|_0^1\,d\theta=2\int_0^{2\pi}\frac{1}{3}\,d\theta=\frac{4\pi}{3}\end{aligned}$$

Ex. Find $\displaystyle\iint_S x\,dS$, where S is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.
Sol. The surface is a piece of a plane $ax+by+cz=d$ and putting in the 3 points we get $a=d$, $b=d$ and $c=d$, e.g. $a=b=c=d=1$. The surface is therefore given by $h(x,y,z)=x+y+z=1$ and $(x,y)\in D=\{(x,y);x\ge0,y\ge0,x+y\le1\}$. The normal to the surface is therefore $\mathbf{n}=\nabla h/|\nabla h|=(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$. We have

$$dS=\frac{dxdy}{|\cos\gamma|}=\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}=\sqrt{3}dxdy$$

If we rewrite $D=\{(x,y);0\le x\le1,\,0\le y\le1-x\}$ we get

$$\iint_S x\,dS=\iint_D x\sqrt{3}\,dxdy=\int_0^1\!\!\int_0^{1-x}x\sqrt{3}\,dy\,dx=\int_0^1 xy\sqrt{3}\Big|_{y=0}^{1-x}dx=\int_0^1 x(1-x)\sqrt{3}\,dx=\frac{\sqrt{3}}{6}$$

A surface is called **closed** if it has no boundary.
The sphere is closed but the upper hemisphere is not since its boundary is the equator. A closed surface has an inside and an outside.

An **oriented surface** is a two-sided surface with one side specified as the **outside**.
The **outward** normal points away from from the outside of the surface.
A orientation can hence be specified by giving an outward normal everywhere.
The upper hemisphere is orientable.
A Möbius strip is not orientable since it only has one side.

A surface is called **regular** if it has a tangent plane everywhere. A parameterized surface is regular if $\mathbf{X}_u \times \mathbf{X}_v$ is nonvanishing everywhere.
A sphere is a regular surface but a cone is not regular at the tip.

There are a couple of alternative ways to express the surface area element:

$$dS = \|\mathbf{X}_u \times \mathbf{X}_v\| dudv = \sqrt{\|\mathbf{X}_u\|^2 \|\mathbf{X}_v\|^2 - (\mathbf{X}_u \cdot \mathbf{X}_v)^2}\ dudv$$

and since $\mathbf{X} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ we can also write (after some work)

$$dS = \|\mathbf{X}_u \times \mathbf{X}_v\| dudv = \sqrt{\Big(\frac{\partial(x,y)}{\partial(u,v)}\Big)^2 + \Big(\frac{\partial(y,z)}{\partial(u,v)}\Big)^2 + \Big(\frac{\partial(x,z)}{\partial(u,v)}\Big)^2}\ dudv$$

For a surface of revolution of a function $y = f(x)$ about the x-axis we have

$$\text{Area} = 2\pi \int_a^b |f(x)| \sqrt{1 + (f'(x))^2}\, dx$$

and revolved about the y axis

$$\text{Area} = 2\pi \int_a^b |x| \sqrt{1 + (f'(x))^2}\, dx$$

Surface Integrals of vector functions.
The flow rate of fluid out of the total surface S, or the **flux** of the velocity vector field $\mathbf{F}$ out of the surface S, with outward unit normal $\mathbf{n}$, is given by

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$$

That only the normal component of $\mathbf{F}$ matters is clear since a tangential velocity would not contribute to the flow of fluid out from the surface.

The question of how to calculate the flux reduces how to calculate surface integrals. In a parametrization $\mathbf{X} = \mathbf{X}(u, v)$ we have

$$dS = \|\mathbf{X}_u \times \mathbf{X}_v\|\, dudv, \qquad \mathbf{n} = \pm\mathbf{X}_u \times \mathbf{X}_v/\|\mathbf{X}_u \times \mathbf{X}_v\|.$$

Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \pm \iint \mathbf{F}\cdot(\mathbf{X}_u \times \mathbf{X}_v)\, dudv$$

Here the sign is positive if $\mathbf{X}_u \times \mathbf{X}_v$ points out from the surface.

Ex. Find the flux of $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ out of the surface S of the cube $C = \{(x, y, z);\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$.
Sol. The Cube has six sides S_1 with $x = 0$, S_2 with $x = 1$, S_3 with $y = 0$, S_4 with $y = 1$, S_5 with $z = 0$ and S_6 with $z = 1$. On S_1, the outward normal is $-\mathbf{i}$ and $\mathbf{F}\cdot\mathbf{n} = (y\,\mathbf{j} - 2z\,\mathbf{k})\cdot(-\mathbf{i}) = 0$, on S_2, $\mathbf{F}\cdot\mathbf{n} = (\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k})\cdot\mathbf{i} = 1$, on $\mathbf{S}_3$, $\mathbf{F}\cdot\mathbf{n} = 0$, on S_4, $\mathbf{F}\cdot\mathbf{n} = 1$, on S_5, $\mathbf{F}\cdot\mathbf{n} = 0$, and on S_6, $\mathbf{F}\cdot\mathbf{n} = -2$. Since the area of each side is one it follows that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + ... + \iint_{S_6} \mathbf{F}\cdot\mathbf{n}\, dS = 0 + 1 + 0 + 1 + 0 - 2 = 0$$

Ex. Find the flux of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}$ out of the sphere $S = \{(x,y,z);\, x^2+y^2+z^2 = 1\}$.
Sol. The surface can be written $h(x,y,z) = x^2+y^2+z^2 = 1$. The outward unit normal to the unit sphere is $\mathbf{n} = \nabla h/|\nabla h| = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})/\sqrt{x^2+y^2+z^2} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, when $x^2+y^2+z^2 = 1$. Therefore

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S x^2+y^2-2z^2\,dS$$

There are several ways to proceed:

(1) In Spherical coordinates, $\mathbf{X}(\phi,\theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$. Then $dS = |\mathbf{X}_\phi \times \mathbf{X}_\theta|\,d\phi d\theta = \sin\phi\,d\phi d\theta$. Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\!\!\int_0^{\pi}\big(\sin^2\!\phi\cos^2\!\theta + \sin^2\!\phi\sin^2\!\theta - 2\cos^2\!\phi\big)\sin\phi\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\!\!\int_0^{\pi}(\sin^2\!\phi - 2\cos^2\!\phi)\sin\phi\,d\phi d\theta = \int_0^{2\pi}\!\!\int_0^{\pi}(1-3\cos^2\!\phi)\sin\phi\,d\phi d\theta$$
$$= \int_0^{2\pi} -\cos\phi + \cos^3\!\phi\Big|_{\phi=0}^{\pi}\,d\theta = 0$$

(2) The sphere S can be written as the union of the northern hemisphere S_+ and southern hemisphere S_+ and each of these can be viewed as a graph over the x-y plane $S_\pm = \{(x,y,z);\, z = \sqrt{1-x^2-y^2},\, (x,y)\in D\}$, where $D = \{(x,y);\, x^2+y^2 \le 1\}$. Since the integrand and the sphere are symmetric under changing z to $-z$ we have

$$\iint_S x^2+y^2-2z^2\,dS = 2\iint_{S_+} x^2+y^2-2z^2\,dS$$

We can now instead write $dS = \dfrac{dxdy}{|\cos\gamma|} = \dfrac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \dfrac{dxdy}{z} = \dfrac{dxdy}{\sqrt{1-x^2-y^2}}$ so

$$\iint_{S_+} x^2+y^2-2z^2 dS = \iint_D \big(3(x^2+y^2)-2\big)\frac{dxdy}{\sqrt{1-x^2-y^2}}$$

Introducing polar coordinates we get

$$\iint_D \frac{dxdy}{\sqrt{1-x^2-y^2}} = \int_0^{2\pi}\int_0^1 (3r^2-2)(1-r^2)^{-1/2}r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 -3(1-r^2)^{1/2}r + (1-r^2)^{-1/2}\,dr\,d\theta = \int_0^{2\pi}(1-r^2)^{3/2} - (1-r^2)^{1/2}\Big|_{r=0}^{1} d\theta = 0.$$

(3) Finally, one can also use symmetry to see that

$$\iint_S x^2 dS = \iint_S y^2\,dS = \iint_S z^2\,dS.$$