Lecture 17: 7.1 Parameterized surface. A **Parameterized surface** is given in terms of two parameters

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \qquad \text{or} \qquad \mathbf{X}(u, v) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

A particular example of a parameterized surface is a graph:

$$z = f(x, y), \qquad \text{or} \qquad \mathbf{X}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}$$

Ex The sphere $x^2 + y^2 + z^2 = r^2$ can be parameterized using spherical coordinates:

$$x = r \sin\phi \cos\theta, \quad x = r \sin\phi \sin\theta, \quad z = r \cos\phi, \qquad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi$$

It can however, not be written as one graph, but one for the southern hemisphere $z = -\sqrt{r^2 - x^2 - y^2}$ and one for the northern hemisphere $z = \sqrt{r^2 - x^2 - y^2}$.

A surface is locally close to its tangent plane which is determined by its normal that we now will find. Another description of a surface is a level surface

$$h(x, y, z) = 0, \qquad \text{if} \qquad \nabla h(x, y, z) \neq \mathbf{0}.$$

(The graph is a special case with $h(x, y, z) = z - f(x, y)$.) In this case a normal is

$$\mathbf{N} = \nabla h$$

Ex The sphere $h(x, y, z) = x^2 + y^2 + z^2 = r^2$ with norm $\mathbf{N} = \nabla h = 2\big(x\mathbf{i} + y\mathbf{k} + z\mathbf{k}\big)$.

In particular the usual way to describe a plane is as a level surface:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

where $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a normal to the plane and $\mathbf{c}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ is a point in the plane. However we can also give the **parametric equations of a plane**

$$\mathbf{X}(u, v) = \mathbf{A}u + \mathbf{B}v + \mathbf{c}_0,$$

where $\mathbf{A}$ and $\mathbf{B}$ are vectors in the plane. To go from the parametric equations to the usual equation we note that the normal to the plane is given by $\mathbf{N} = \mathbf{A} \times \mathbf{B}$.

The function $u \to \mathbf{X}(u, v)$, where v is kept constant and u vary, is a parameterized curve and

$$\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

is tangent to this curve, and hence to the surface. Similarly the vector

$$\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

is tangent to the curves $v \to \mathbf{X}(u, v)$, and hence to the surface. The tangent plane to the surface is spanned by $\mathbf{X}_u$ and $\mathbf{X}_v$ so a normal to the surface is given by

$$\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v$$

Ex 2 Find a normal to the surface $x = u\cos v$, $y = u \sin v$, $z = u$.
Sol $\mathbf{X} = u\cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + u\,\mathbf{k}$ so $\mathbf{X}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}$, $\mathbf{X}_v = -u \sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$.
$\mathbf{N} = \mathbf{X}_u \times \mathbf{X}_v = \cdots = -u\cos v\,\mathbf{i} - u \sin v\,\mathbf{j} + u\,\mathbf{k}$.

Area of a parameterized surface.

For $(u,v) \sim (u_i, v_j)$ the surface is close to its tangent plane at the point (u_i, v_j) and $\mathbf{X}$ is close to the **linear approximation**:

$$\mathbf{X}(u,v) \sim \mathbf{L}_{ij}(u,v) = \mathbf{X}(u_i,v_j) + \mathbf{X}_u(u_i,v_j)(u-u_i) + \mathbf{X}_v(u_i,v_j)(v-v_j)$$

The image $S_{ij} = \mathbf{X}(R^*_{ij})$ of a small rectangle

$$R^*_{ij} = \{(u,v);\ u_i \le u \le u_i + \triangle u,\ \ v_j \le v \le v_j + \triangle v\}.$$

under the map $\mathbf{X}$ is close to the image under the linear map $\mathbf{L}_{ij}$. The image of R^*_{ij} under $\mathbf{L}_{ij}$ is a parallelogram with adjacent sides $\mathbf{X}_u \triangle u$ and $\mathbf{X}_v \triangle v$ so

$$\text{Area}\,(S_{ij}) \sim \|\mathbf{X}_u \times \mathbf{X}_v\|\, \triangle u \triangle v = \|\mathbf{X}_u \times \mathbf{X}_v\|\, \text{Area}\,(R^*_{ij})$$

Summing up over all small rectangles in the u-v plane we get

$$\text{Area}\,(S) = \sum \text{Area}\,(S_{ij}) \sim \sum \|\mathbf{X}_u \times \mathbf{X}_v(u_i,v_j)\| \triangle u \triangle v$$

and in the limit as $\triangle u, \triangle v \to 0$ we get the formula for the **surface area** of a parameterized surface:

$$\text{Area}\,(S) = \iint \|\mathbf{X}_u \times \mathbf{X}_v(u,v)\|\, dudv$$

This is in fact invariant under parameterizations.

Ex. Find the area of the sphere S of radius r.
Sol. Using the parametrization $\mathbf{X} = r\sin\phi\cos\theta\,\mathbf{i} + r\sin\phi\sin\theta\,\mathbf{j} + r\cos\phi\,\mathbf{k}$ we get $\mathbf{X}_\phi = r\cos\phi\cos\theta\,\mathbf{i} + r\cos\phi\sin\theta\,\mathbf{j} - r\sin\phi\mathbf{k}$ and $\mathbf{X}_\theta = -r\sin\phi\sin\theta\mathbf{i} + r\sin\phi\cos\theta\mathbf{j}$;

$$\mathbf{X}_\phi \times \mathbf{X}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{vmatrix}$$
$$= r^2\sin^2\phi\cos\theta\,\mathbf{i} + r^2\sin^2\phi\sin\theta\,\mathbf{j} + r^2\sin\phi\cos\phi\,\mathbf{k}$$

and $|\mathbf{X}_\phi \times \mathbf{X}_\theta| = r^2|\sin\phi|\sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi} = r^2\sin\phi$. Hence $\text{Area}\,(S) = \int_0^{2\pi}\int_0^{\pi} r^2\sin\phi\, d\phi d\theta = \int_0^{2\pi} -r^2\cos\phi\big|_0^{\pi}\, d\theta = \int_0^{2\pi} 2r^2\, d\theta = 4\pi r^2.$

Surface area of a graph. In the special case of a graph $z = f(x,y)$ we have $\mathbf{X}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + f(x,y)\,\mathbf{k}$ we have $\mathbf{X}_x = \mathbf{i} + f_x(x,y)\,\mathbf{k}$, $\mathbf{X}_y = \mathbf{j} + f_y(x,y)\,\mathbf{k}$ and

$$\mathbf{X}_x \times \mathbf{X}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}$$

and

$$\|\mathbf{X}_x \times \mathbf{X}_y\| = \sqrt{1 + f_x^2 + f_y^2}$$

and hence we get the formula for the area of a graph:

$$\text{Area}\,(S) = \iint \sqrt{1 + f_x^2 + f_y^2}\;dxdy$$

There is however a simpler way to remember this formula. Let

$$R_{ij}^* = \{(x,y);\, x_i \le x \le x_i + \triangle x,\, y_j \le y \le y_j + \triangle y\},$$

be a small rectangle in the x-y plane. Above this rectangle is a parallelogram in the tangent plane to the surface at $(x_i, y_j, f(x_i, y_j))$, that projects down to the rectangle in the x-y plane. The quotient of the area of the rectangle in the x-y plane to the area of the parallelogram in the tangent plane above it is the cosine of the angle γ between the tangent plane and the x-y plane. Hence

$$\text{Area}\,(S) = \iint \frac{dxdy}{|\cos\gamma|}$$

If $\mathbf{n}$ is the unit normal to the tangent plane and $\mathbf{k}$ is the normal to the x-y plane then the angle is given by $\cos\gamma = \mathbf{n}\cdot\mathbf{k}$.

The unit normal to a graph $z = f(x,y)$ is easiest calculated by writing it in the form $h(x,y,z) = z - f(x,y) = 0$:

$$\mathbf{n} = \frac{\nabla h}{\|\nabla h\|} = \frac{-f_x\,\mathbf{i} - f_y\,\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$$

If we also take the inner product with $\mathbf{k}$ the desired formula follows since

$$|\cos\gamma| = |\mathbf{n}\cdot\mathbf{k}| = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}$$

Ex Find the area of the part of the cone $S = \{(x,y,z);\, z = \sqrt{x^2+y^2},\, x^2+y^2 \le 1\}$.
Sol The surface is a graph and the angle between the surface and the $x - y$ plane is $\gamma = \pi/4$, since when say $y = 0$ its just $z = |x|$. Hence $\cos\gamma = 1/\sqrt{2}$. Alternatively one can calculate $|\mathbf{n}\cdot\mathbf{k}|$. Hence

$$\text{Area}(S) = \int_{x^2+y^2\le 1} \frac{dxdy}{|\cos\gamma|} = \sqrt{2}\int_{x^2+y^2\le 1} dxdy = \sqrt{2} \times \text{Area of unit disc} = \sqrt{2}\pi.$$