Lecture 16: 6.2 Green's theorem. Suppose that D is a domain in the plane with boundary curve C going in positive direction, i.e. walking in the direction of C the domain D should be on your left. **Green's theorem** says that

$$\int_C P dx + Q dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dxdy$$

This is somewhat similar to the first fundamental theorem of calculus:

$$f(b) - f(a) = \int_a^b f'(x)\, dx.$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ then the left of Green's theorem is $\int_C \mathbf{F}\cdot d\mathbf{s}$, and $\mathbf{curl}\,\mathbf{F} = \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathbf{k}$

Ex Let $P = xy$, $Q = y^2$ and $D = \{(x,y);\, 0 \le x \le 1,\, x^2 \le y \le x\}$. Let C be the positively oriented boundary of D. Calculate both sides of Green's theorem.
Sol. We have

$$\iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dxdy = \iint_D -x\, dxdy = \int_0^1 \int_{x^2}^x -x\, dy\, dx = \int_0^1 -x(x - x^2)\, dx$$

$$= -\frac{x^3}{3} + \frac{x^4}{4}\Big|_0^1 = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}.$$

The boundary consists of two parts C_1; $x = t$, $y = t^2$, $0 \le t \le 1$ and C_2; $x = 1 - t$, $y = 1 - t$, $0 \le t \le 1$. Note that C_2 is oriented so it starts at $(x,y) = (1,1)$ in order that the total curve should be positively oriented. Let $P = xy$ and $Q = y^2$.

$$\int_C P\, dx + Q\, dy = \int_{C_1} \Big(xy\frac{dx}{dt} + y^2\frac{dy}{dt}\Big)\, dt + \int_{C_2} \Big(xy\frac{dx}{dt} + y^2\frac{dy}{dt}\Big)\, dt$$

$$= \int_0^1 t^3 + 2t^5\, dt + \int_0^1 -(1-t)^2 - (1-t)^2\, dt = \frac{t^4}{4} + \frac{t^6}{3}\Big|_0^1 + \frac{2(1-t)^3}{3}\Big|_0^1 = \frac{1}{4} + \frac{1}{3} - \frac{2}{3} = -\frac{1}{12}.$$

If $(P,Q) = (0,x)$ and $(P,Q) = (-y,0)$ we get

$$\text{Area}\,(D) = \iint_D 1\, dxdy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dx$$

From Green's theorem applied to $(P,Q) = (0,x)$, $(P,Q) = (-y,0)$ respectively $(P,Q) = (-y/2, x/2)$

$$\text{Area}\,(D) = \int_C x\, dy = -\int_C y\, dx = \frac{1}{2}\int_C -y\, dx + x\, dy$$

which gives another way to calculate the area.

Ex. Find the area of the interior of an ellipse: $D = \{(x,y);\, (x/a)^2 + (y/b)^2 \le 1\}$
Sol. Parameterizing the ellipse $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$ and using (5.4.2)

$$\text{Area}\,(D) = \frac{1}{2}\int_C -y\, dx + x\, dy = \frac{1}{2}\int_0^{2\pi} -y\frac{dx}{dt} + x\frac{dy}{dt}\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} -b\sin t(-a\sin t) + (a\cos t)b\sin t\, dt = \frac{1}{2}\int_0^{2\pi} ab(\sin^2 t + \cos^2 t)\, dt = \pi ab$$

Proof of Green's theorem. The proof reduces to proving the two equalities

$$\int_C P dx = \iint_D -\frac{\partial P}{\partial y} dxdy, \qquad \text{and} \qquad \int_C Q dy = \iint_D \frac{\partial Q}{\partial x} dxdy,$$

since the theorem has to be true separately for the cases $(P, 0)$ and $(0, Q)$. Let us just prove the first equality for P since the proof for the second is similar.

Suppose D is a region of type I: $D = \{(x, y);\, a \le x \le b,\, f_1(x) \le y \le f_2(x)\}$. By the Fundamental Theorem of Calculus

$$-\iint_D \frac{\partial P}{\partial y} dxdy = -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y} dy\, dx = \int_a^b -P(x, f_2(x))\, dx + \int_a^b P(x, f_1(x))\, dx$$

The boundary C of D consists of four parts, a bottom C_1 where $y = f_1(x)$, a right side C_2 where $x = b$, a top C_3 where $y = f_2(x)$, a left side C_4 where $x = a$. Each of these four curves have to be positively oriented.

At the bottom C_1 we have $x = t$, $y = f_1(t)$, $a \le t \le b$:

$$\int_{C_1} P\, dx = \int_a^b P(t, f_1(t))\, \frac{dx}{dt} dt = \int_a^b P(x, f_1(x))\, dx$$

On the right side we have $x = b$, $y = t$, $f_1(b) \le t \le f_2(t)$, so $\frac{dx}{dt} = 0$ so $\int_{C_2} Pdx = 0$. The top C_3 is oriented to go backwards from $x = b$ to $x = 2a$. The integral over C_3 is therefor $-$ the integral over the curve C_3^- going in the reverse direction $x = t$, $y = f_2(t)$, $a \le t \le b$:

$$\int_{C_3} P\, dx = -\int_{C_3^-} P\, dx = -\int_a^b P(t, f_2(t))\, \frac{dx}{dt} dt = -\int_a^b P(x, f_2(x))\, dx$$

The integral over C_4 also vanishes as above. Adding up the contributions from the different curves proves the theorem for P in case D is a region of type I.

If D is not a region of type I we can divide D up into regions D_1,...,D_N of type I. The right hand side integral over D in Green's Theorem is then just sum of the integrals over the regions D_1,...,D_N. Similarly the integral over C is the sum of the integrals over their boundaries C_1,...,C_N. This is because the segments of the boundary curves that belong to two curves go in the opposite direction for the two curves so they integrals over those parts of the curves cancel each other.

6.3 Conservative Fields.

A vector field is called **gradient** if it is a gradient $\mathbf{F} = \mathbf{grad}\ \phi$ of a scalar **potential**.

It is called **path independent** if the line integral depends only on the endpoints, i.e. if $\mathbf{c}_1$ and $\mathbf{c}_2$ are any two paths from P to Q then $\displaystyle\int_{\mathbf{c}_1} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}_2} \mathbf{F} \cdot d\mathbf{s}$.
This is equivalent to that the line integral along any closed path or loop vanishes.

Th A vector field $\mathbf{F}$ in a domain D is gradient if and only if it is path independent.

In that case we say that it is **conservative** and the integral is the difference in potential of the endpoint minus initial point: $\displaystyle\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \phi(Q) - \phi(P)$.

Note that $\mathbf{F} = \mathbf{grad}\ \phi$ is perpendicular to the level surfaces of ϕ and hence the level surfaces of the potential function are perpendicular to the flow lines of $\mathbf{F}$.

Proof of Th: If $\mathbf{F} = \mathbf{grad}\ \phi$ then for any curve $\mathbf{c}$ from P to Q:

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}\, dt = \int_a^b \frac{d}{dt}\phi(x(t), y(t), z(t))\, dt = \phi(Q) - \phi(P),$$

If the integral is independent of the way then we define

$$\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{s},$$

where the integral is along any curve from a fixed point (x_0, y_0, z_0) to (x, y, z).
This is well-defined since it does not depend on which path we integrate along.
In particular we can pick a curve going from (x_0, y_0, z_0) to (a, y, z) and then along a straight line segment (t, y, z), $a \le t \le x$, to (x, y, z):

$$\phi(x, y, z) = \int_{(x_0,y_0,z_0)}^{(a,y,z)} \mathbf{F} \cdot d\mathbf{s} + \int_{(a,y,z)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{s} = \int_{(x_0,y_0,z_0)}^{(a,y,z)} \mathbf{F} \cdot d\mathbf{s} + \int_a^x F_1(t, y, z) dt.$$

The first integral is independent of x and by the fundamental theorem of calculus

$$\frac{\partial \phi}{\partial x}(x, y, z) = \frac{\partial}{\partial x}\int_a^x F_1(t, y, z) dt = F_1(x, y, z).$$

The proof of that $\partial\phi/\partial y = F_2$ and $\partial\phi/\partial z = F_3$ is similar.

Conservative fields-Irrotational fields. We have seen an example of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ that could not be conservative because if there was a potential $\mathbf{F} = \mathbf{grad}\ \phi$ then we must have $\partial F_1/\partial y = \partial F_2/\partial x$ since $\partial^2\phi/\partial x\partial y = \partial^2\phi/\partial y\partial x$. Hence for a vector field to be conservative we must have

$$\mathbf{curl}\,\mathbf{F} = \Big(\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y}\Big)\mathbf{i} + \Big(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\Big)\mathbf{j} + \Big(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x}\Big)\mathbf{k} = \mathbf{0}$$

A vector field satisfying this is called **irrotational**. Is it sufficient that a vector field is irrotational for it to be conservative? To answer that question we introduce:
Def A domain D is called **simply-connected** if every closed curve C in D can be continuously shrunk to a point in D without leaving D.

Theorem. *A vector field* $\mathbf{F}$ *defined and continuously differentiable throughout a simply connected domain* D *is conservative if and only if it is irrotational in* D.

The physical interpretation of this is that the flow lines for a gradient vector field can not curl around in a closed orbit since ϕ increases in the direction of the gradient.

Ex. Show that $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ is conservative in all space.
Sol. By the previous theorem it suffices to show that it is irrotational: $\mathbf{curl}\,\mathbf{F}=\mathbf{0}$.

Ex. Is $\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2+y^2)$ conservative in $D=\{(x,y,z);\ (x,y)\neq(0,0)\}$.
Sol. $\nabla\times\mathbf{F} = ... = \mathbf{0}$ but D is not simply connected so it does not follow that it is conservative. In fact, it is not since the line line integral along a circle around the z-axis is non vanishing as we shall see. If $x=\cos t$, $y=\sin t$ and $z=0$, $0\le t\le 2\pi$ then $\int_C \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi}(F_1dx/dt + F_2dy/dt + F_3dz/dt)dt = \int_0^{2\pi}\sin^2 t + \cos^2 t\, dt = 2\pi$.

Proof of the theorem. That conservative implies irrotational is just the calculation above that $\nabla\times\nabla\phi = 0$. We shall prove that irrotational implies conservative if the domain is all of space or a rectangular box containing the origin. We define

$$\phi(x,y,z) = \int_0^x F_1(t,0,0)\,dt + \int_0^y F_2(x,t,0)\,dt + \int_0^z F_3(x,y,t)\,dt$$

and we will show that $\nabla\phi=\mathbf{F}$ if $\nabla\times\mathbf{F}=0$. By the Fundamental Theorem of Calculus

$$\frac{\partial\phi}{\partial z}(x,y,z) = \frac{d}{dz}\Big|_{(x,y)-fixed}\phi(x,y,z) = \frac{d}{dz}\int_0^z F_3(x,y,t)\,dt = F_3(x,y,z)$$

Also using that we can differentiate below an integral sign and $\partial F_3/\partial y = \partial F_2/\partial z$;

$$\begin{aligned}\frac{\partial\phi}{\partial y}(x,y,z) &= F_2(x,y,0) + \int_0^z \frac{\partial F_3}{\partial y}(x,y,t)\,dt = F_2(x,y,0) + \int_0^z \frac{d}{dt}F_2(x,y,t)\,dt\\ &= F_2(x,y,0) + F_2(x,y,z) - F_2(x,y,0) = F_2(x,y,z)\end{aligned}$$

Finally by the same argument

$$\begin{aligned}\frac{\partial\phi}{\partial x}(x,y,z) &= F_1(x,0,0) + \int_0^y \frac{\partial F_2}{\partial x}(x,t,0)\,dt + \int_0^z \frac{\partial F_3}{\partial x}(x,y,t)\,dt\\ &= F_1(x,0,0) + \int_0^y \frac{d}{dt}F_1(x,t,0)\,dt + \int_0^z \frac{d}{dt}F_1(x,y,t)\,dt\\ &= F_1(x,0,0) + F_1(x,y,0) - F_1(x,0,0) + F_1(x,y,z) - F_1(x,y,0) = F_1(x,y,z)\end{aligned}$$