Lecture 15: 5.5 Changing variables in Triple Integrals. If $\mathbf{T}(u,v,w) = (x,y,z)$ is an invertible mapping $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ and $W = T(W^*)$ then

$$\iiint_W f(x,y,z)\,dxdydz = \iiint_{W^*} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw,$$

where the Jacobian determinant is

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

Symbolically:

$$dxdydz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

In **spherical coordinates** $x = r\sin\phi\cos\theta$, $y = r\sin\phi\sin\theta$, $z = r\cos\phi$, $r \geq 0$, $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$, we have $dxdydz = \dfrac{\partial(x,y,z)}{\partial(r,\phi,\theta)} dr\,d\phi\,d\theta = \cdots = r^2 \sin\phi\,dr\,d\phi\,d\theta$

Ex. Find the volume of the unit ball $B = \{(x,y,z);\ x^2 + y^2 + z^2 \leq 1\}$.

Sol.1 $$\iiint_B 1\,dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2 \sin\phi\,dr\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{r^3}{3}\Big|_0^1 \sin\phi\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} \frac{\sin\phi}{3}\,d\phi\,d\theta = \int_0^{2\pi} -\frac{\cos\phi}{3}\Big|_0^{\pi} d\theta \int_0^{2\pi} \frac{2}{3}\,d\theta = \frac{4\pi}{3}.$$

6.1 Line Integrals. Recall the definition of arc-length of a parameterized curve $\mathbf{c}(t)$, $a \leq t \leq b$. One starts by dividing the curve up into smaller curves, $a = t_0 < t_1 < ... < t_n = b$, $t_k = a + k\triangle t$, $\triangle t = (b-a)/n$, with endpoints $\mathbf{c}_k = \mathbf{c}(t_k)$. Then

$$\triangle \mathbf{c}_k = \mathbf{c}(t_k + \triangle t) - \mathbf{c}(t_k) \sim \mathbf{c}'(t_k)\triangle t$$

The **arc-length** is then is then given by

$$\int_{\mathbf{c}} ds = \lim_{n\to\infty} \sum_{k=1}^{n} \|\triangle \mathbf{c}_k\| = \lim_{n\to\infty} \sum_{k=1}^{n} \|\mathbf{c}'(t_k)\| \triangle t = \int_a^b \|\mathbf{c}'(t)\|\, dt$$

The **path integral** of a function f over a curve C is defined by

$$\int_{\mathbf{c}} f\, ds = \int_a^b f(\mathbf{c}(t))\, \|\mathbf{c}'(t)\|\, dt$$

If the curve is in the x-y plan it can be interpreted as the area of the surface in space formed by going straight up from the curve to the graph of the function $z = f(x, y)$.

Before we define the **line integral** let us give its **physical motivation**. Suppose that a force $\mathbf{F}$ is acting on a particle as it moves. Suppose first that the force is constant and the particle moves along a straight line. Let $\mathbf{d}$ be the displacement vector from the initial to the final position. The **work done by the force** on the particle is

$$W = \mathbf{F} \cdot \mathbf{d}$$

If the force is a vector field $\mathbf{F}(\mathbf{x})$ that acts on a particle tracing out a curve $\mathbf{c}(t)$ then the work done by the force moving the particle from $\mathbf{c}_{k+1}$ to $\mathbf{c}_k$ is approximately

$$\triangle W_k \sim \mathbf{F}(\mathbf{c}_k) \cdot \triangle \mathbf{c}_k \sim \mathbf{F}(\mathbf{c}(t_k)) \cdot \mathbf{c}'(t_k)\, \triangle t$$

Hence the total work done on the particle is

$$W = \lim_{n\to\infty} \sum_{k=1}^{n} \triangle W_k = \lim_{n\to\infty} \sum_{k=1}^{n} \mathbf{F}(\mathbf{c}(t_k)) \cdot \mathbf{c}'(t_k)\, \triangle t = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

We therefore define the **line integral** of a vector field $\mathbf{F}$ over the curve $\mathbf{c}$ to be

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

This can be written as the path integral of the tangential component of the force

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \frac{\mathbf{c}'(t)}{\|\mathbf{c}(t)\|} \|\mathbf{c}'(t)\|\, dt = \int_{\mathbf{c}} \mathbf{F} \cdot \mathbf{T}\, ds, \quad \text{where} \quad \mathbf{T} = \frac{\mathbf{c}'(t)}{\|\mathbf{c}(t)\|}$$

Another way to write the line integral is

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} F_1\, dx + F_2\, dy + F_3\, dz$$

The meaning of this is however just

$$\int_a^b \Big(F_1(x(t), y(t), z(t))\, \frac{dx}{dt} + F_2(x(t), y(t), z(t)) \frac{dy}{dt} + F_3(x(t), y(t), z(t)) \frac{dz}{dt} \Big) dt$$

which follows since $\mathbf{F} \cdot \mathbf{c}' = (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \cdot (x'\, \mathbf{i} + y'\, \mathbf{j} + z'\, \mathbf{k}) = F_1 x' + F_2 y' + F_3 z'$.

Ex. 1 Evaluate $\displaystyle\int_{\mathbf{c}} x^2 dx + xy dy + dz$ where $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$, $0 \leq t \leq 1$. **Sol.**

$$\int_0^1 \Big(x^2 \frac{dx}{dt} + xy \frac{dy}{dt} + \frac{dz}{dt} \Big) dt = \int_0^1 (t^2 + 2t^4) dt = \frac{1}{3} t^3 + \frac{2}{5} t^5 \Big|_0^1 = \frac{11}{15}$$

Examples

Ex. 1 Evaluate $\int_{\mathbf{c}} x^2dx + xydy + dz$ where $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}$, $0 \le t \le 1$. **Sol.**

$$\int_0^1 \Big(x^2\frac{dx}{dt} + xy\frac{dy}{dt} + \frac{dz}{dt}\Big)dt = \int_0^1 (t^2 + 2t^4)dt = \frac{1}{3}t^3 + \frac{2}{5}t^5\Big|_0^1 = \frac{11}{15}$$

A **reparametrization** of the curve $\mathbf{c} : [a_1, b_1] \to \mathbf{R}^3$ is a curve $\mathbf{p} = \mathbf{c}\circ h : [a, b] \to \mathbf{R}^3$, where $h : [a, b] \to [a_1, b_1]$ is an invertible map. If $h(a) = a_1$ and $h(b) = b_1$ then we say that it is **orientation preserving**. If $h(a) = b_1$ and $h(b) = a_1$ then we say that it is **orientation reversing**. We have:

Th(Change of parametrization) If $\mathbf{p}$ is a reparmetrization of $\mathbf{c}$ then

$$\int_{\mathbf{p}} \mathbf{F}\cdot d\mathbf{s} = \pm \int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s}$$

if it is orientation preserving $(+)$ and if it is orientation reversing $(-)$.

Ex 2 Evaluate the integral in Ex 1 $\int_{\mathbf{p_i}} x^2dx + xydy + dz$ if

(i) $\mathbf{p}_1(t) = \mathbf{c}(t^2) = t^2\mathbf{i} + t^4\mathbf{j} + \mathbf{k}$, $0 \le t \le 1$.
(ii) $\mathbf{p}_2(t) = \mathbf{c}(1-t) = (1-t)\mathbf{i} + (1-t)^2\mathbf{j} + \mathbf{k}$, $0 \le t \le 1$.

Sol.

$$(i) \quad \int_0^1 \Big(x^2\frac{dx}{dt} + xy\frac{dy}{dt} + \frac{dz}{dt}\Big)dt = \int_0^1 (2t^5 + 4t^9)dt = \frac{1}{3}t^6 + \frac{2}{5}t^{10}\Big|_0^1 = \frac{11}{15},$$

$$(ii) \quad \int_0^1 \Big(x^2\frac{dx}{dt} + xy\frac{dy}{dt} + \frac{dz}{dt}\Big)dt = \int_0^1 (-(1-t)^2 - 2(1-t)^4)dt = \cdots = -\frac{11}{15}.$$

Ex 3a Evaluate $\int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$ and $\mathbf{c}_1(t) = (1-t)\mathbf{i}$, $0 \le t \le 1$ and $\mathbf{c}_1(t) = (t-1)\mathbf{i} + (t-1)\mathbf{j}$, when $1 \le t \le 1 + 1/\sqrt{2}$.

Sol.: We divide $\mathbf{c}_1$ up into two parts. When $0 \le t \le 1$ then $x = (1-t)$ and $y = 0$ so $dx/dt = -1$ and $dy/dt = 0$ and when $1 \le t \le 1 + 1/\sqrt{2}$ we have $x = (t-1)$, $y = (t-1)$ and $dx/dt = dy/dt = 1$ so

$$\int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s} = \int_0^1 \Big(y\frac{dx}{dt} + x\frac{dy}{dt}\Big)\,dt + \int_1^{1/\sqrt{2}} \Big(y\frac{dx}{dt} + x\frac{dy}{dt}\Big)\,dt$$
$$= \int_0^1 0\,dt + \int_1^{1/\sqrt{2}} \big((t-1)1 + (t-1)\big)\,dt = (t-1)^2\big|_1^{1+1/\sqrt{2}} = \frac{1}{2}$$

Ex 3b Evaluate $\int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F} = y\mathbf{i}+x\mathbf{j}$ and $\mathbf{c}_2(t) = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}$, $0 \le t \le \pi/4$.

Sol On $\mathbf{c}_2$ we have $x = \cos t$, $y = \sin t$, $dx/dt = -\sin t$, $dy/dt = \cos t$ so

$$\int_{\mathbf{c}_2}\mathbf{F}\cdot d\mathbf{s} = \int_0^{\pi/4}\Big(y\frac{dx}{dt}+x\frac{dy}{dt}\Big)\,dt = \int_0^{\pi/4}(-\sin^2 t+\cos^2 t)\,dt = \int_0^{\pi/4}\cos(2t)\,dt = \frac{\sin(2t)}{2}\Big|_0^{\pi/4} = \frac{1}{2}$$

Conclusion Both line integrals go from $(0, 1)$ to $(1/\sqrt{2}, 1/\sqrt{2})$ over different paths. In this case the value of the line integral is independent of the path. Vector fields for which this is true are called conservative and we shall study these more later. In physical terms the work is independent of the way for conservative vector fields.

6.3 Conservative vector fields.

Th(Line integrals of gradient vector fields) If $\mathbf{c} : [a, b] \to \mathbf{R}^3$ then

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

Pf In fact, by the definition and chain rule

$$\int_{\mathbf{c}} \nabla f \cdot d\mathbf{s} = \int_a^b \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_a^b \frac{d}{dt} f(\mathbf{c}(t))\, dt = f(\mathbf{c}(b)) - f(\mathbf{c}(a))$$

A vector field $\mathbf{F}$ is called **conservative** if it has a **potential** f such that $\mathbf{F} = \nabla f$.

Ex. 4 Let $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$. Find a potential function.
Sol. We want to find f so that $\partial f/\partial x = y$, $\partial f/\partial y = x$ and $\partial f/\partial z = 0$. Integration of $\partial f/\partial x = y$ gives $f = xy + g(y, z)$, where g is any function of y and z. With this f it follows that $\partial f/\partial y = x$ and $\partial f/\partial z = 0$ if $g(y, z) = C$ is a constant. Hence $f = xy + C$, for any constant C, satisfies **grad** $f = \mathbf{F}$.

Ex. 5 Let $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$. Evaluate the line integrals in Ex 3 $\int_{\mathbf{c}_i} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{c}_i(t)$ is a curve starting at $\mathbf{c}_i(0) = 0$, and ending at $\mathbf{c}_i(1) = (1/\sqrt{2}, 1/\sqrt{2})$.
Sol By the theorem, using the potential function $\mathbf{F} = \nabla f$, $f = xy$:

$$\int_{\mathbf{c}_i} \mathbf{F} \cdot d\mathbf{s} = f(1/\sqrt{2}, 1/\sqrt{2}, 0) - f(1, 0, 0) = \frac{1}{2}$$

Ex. 6 Show that $\mathbf{F} = xy^2\mathbf{i} + x^3y\mathbf{j}$ is not conservative.
Sol. If $\partial f/\partial x = xy^2$ then $\partial^2 f/\partial y\partial x = 2xy$ but if $\partial f/\partial y = x^3y$ then $\partial^2 f/\partial x\partial y = 3x^2y$ which is a contradiction.

Theorem If $\mathbf{F}$ is conservative then $\nabla \times \mathbf{F} = 0$.
Pf If $\mathbf{F}$ is conservative then $\mathbf{F} = \nabla f$ and a calcualtion shows that $\nabla \times \nabla f = 0$.