Lecture 14: 5.5 The change of variable theorem. Suppose that $x(u)$, $a \le u \le b$ is a change of variables. In order for it to be invertible we assume that $dx(u)/du > 0$, when $a \le u \le b$. Then we can change variables in the integral:

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))\frac{dx}{du}du \tag{1}$$

i.e. symbolically

$$dx = \frac{dx}{du}du.$$

A small change $\triangle u$ gives a small change $\triangle x \sim x'(u)\triangle u$, by the linear approximation.

We will give similar theorem for functions of two variables. Let $T(u,v) = (x,y)$;

$$x = x(u,y), \qquad y = y(u,v), \tag{2}$$

be a **mapping** from a piecewise smooth simply connected domain D^* in the u-v plane **onto** a simply connected domain $D = T(D^*)$ in the x-y plane. We say that a map is **one-to-one** or **invertible** if no two points are mapped to one point, i.e. if $T(u,v) = T(u',v')$ implies that $(u,v) = (u',v')$. In order for the map to be invertible we assume that the **Jacobian determinant**

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \ne 0, \tag{3}$$

is non-vanishing everywhere, see the **Inverse Function Theorem** in section 2.6. The condition that the determinant of the derivative map $\mathbf{D}e\mathbf{T}$ is non-vanishing is equivalent to the invertibility of the linear approximation

$$\mathbf{T}\big(u+\triangle u, v+\triangle v\big) \sim \mathbf{T}(u,v) + \mathbf{T}_u(u,v)\triangle u + \mathbf{T}_v(u,v)\triangle v,$$

where $\mathbf{T}_u = \begin{bmatrix} \partial x/\partial u \\ \partial y/\partial u \end{bmatrix}$ and $\mathbf{T}_v = \begin{bmatrix} \partial x/\partial v \\ \partial y/\partial v \end{bmatrix}$. In fact the derivative map

$$\mathbf{DT}(\triangle u, \triangle v) = \mathbf{T}_u\triangle u + \mathbf{T}_v\triangle v,$$

is invertible if and only if the column vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ are not parallel.

Change of variable theorem in the plane.

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \tag{4}$$

If $f=1$ then the left is the area of the domain D. We will argue that the Jacobian gives the local change of area scale under the mapping, symbolically

$$dxdy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \tag{5}$$

The geometry of maps of the plane To make sense of (5) we study a linear map:

$$(6) \qquad T(u,v)=(x,y), \quad \text{where} \quad \begin{cases} x = au+bv, \\ y = cu+dv, \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} u + \begin{bmatrix} b \\ d \end{bmatrix} v$$

for which the derivative is

$$\mathbf{DT} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = A, \quad \text{where} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the Jacobian determinant is a constant

$$(7) \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc = \det A.$$

Let R^* be a small rectangle in the u-v plane with side lengths $\triangle u$ and $\triangle v$

$$(8) \qquad R^* = \{(u,v);\, 0 \le u \le \triangle u,\, 0 \le v \le \triangle v\}$$

The image of R^* in the x-y plane

$$R = \{(x,y);\, (x,y)=T(u,v);\, 0 \le u \le \triangle u,\, 0 \le v \le \triangle v\}$$

is a small parallelogram with adjacent sides formed by the two vectors

$$\begin{bmatrix} a \\ c \end{bmatrix} \triangle u, \quad \text{and} \quad \begin{bmatrix} b \\ d \end{bmatrix} \triangle v.$$

Recall that the area is given by the magnitude of the crossproduct of the vectors in space representing adjacent sides: $\mathbf{A} = (a\mathbf{i}+c\mathbf{j})\triangle u + 0\mathbf{k}$ and $\mathbf{B} = (b\mathbf{i}+d\mathbf{j})\triangle u + 0\mathbf{k}$:

$$(10) \qquad \mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\triangle u & c\triangle u & 0 \\ b\triangle v & d\triangle v & 0 \end{vmatrix} = \mathbf{k} \begin{vmatrix} a\triangle u & c\triangle u \\ b\triangle v & d\triangle v \end{vmatrix} = \ldots = \mathbf{k} \begin{vmatrix} a & b \\ c & d \end{vmatrix} \triangle u \triangle v$$

Hence we have proven that for a linear map (6)

$$(11) \qquad \text{Area}\,(R) = \Big| \begin{vmatrix} a & b \\ c & d \end{vmatrix} \Big| \triangle u \triangle v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \text{Area}\,(R^*)$$

We remark that when the determinant (7) vanishes then the area of R vanishes. The map $(u,v)\to(x,y)$ given by (6) is not invertible if the determinant (7) vanishes. If the determinant vanishes then for each (x,y) there are either no solutions (u,v) to (6) or infinitely many. In fact, then the two vectors $\mathbf{A}$ and $\mathbf{B}$ above are parallel and the image R is just a line segment so we can only solve (6) if (x,y) is on this line and in then there are infinitely many solutions since each point on the line can be written in many ways as $u\mathbf{A}+v\mathbf{B}$, if $\mathbf{A}$ and $\mathbf{B}$ are parallel.

Th If T is a linear map with matrix A, $D = T(D^*)$ then Area$(D) = \det A\,$Area(D^*).

In fact, we can divide up the region D^* into small rectangles and their images divides the region D into small parallelograms and the area is the sum of their areas. This proves the change of variable theorem for linear maps.
All maps can be approximated by a linear map close to a point. In the limit as the size of the rectangle R^* tends to 0 the ratio of the areas is given by (5).

The approximate proof of the change of variable theorem.
Let $T(u,v) = (x,y)$ be a mapping $x = x(u,v)$, $y = y(u,v)$, from a piecewise smooth simply connected domain D^* in the u-v plane onto a simply connected domain $D = T(D^*)$ in the x-y plane. Suppose that the **Jacobian determinant**:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

is non-vanishing. Then the **change of variable theorem** states that

$$\iint_D f(x,y)\,dxdy = \iint_{D^*} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv.$$

We proved this for a linear map if $f=1$ when it says that the area of D is the area of D^* times the Jacobian determinant which is the determinant of the linear map. The general case follows by dividing up D^* into smaller sets on which we can approximate the map by its linearization. If (u,v) is close to (u_i,v_j) then $\mathbf{T}(u,v)$ is by Taylor's theorem **approximated by the linear function**

$$\mathbf{T}(u,v) \sim \mathbf{H}(u,v) \equiv \mathbf{T}(u_i,v_j) + \mathbf{T}_u(u_i,v_j)(u-u_i) + \mathbf{T}_v(u_i,v_j)(v-v_j)$$

where

$$\mathbf{T} = x\mathbf{i} + y\mathbf{j} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{T}_u = \frac{\partial \mathbf{T}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial x}{\partial v} \end{bmatrix}, \quad \mathbf{T}_v = \frac{\partial \mathbf{T}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial x}{\partial v} \end{bmatrix}$$

The Jacobian is non-vanishing if and only $\mathbf{T}_u$ and $\mathbf{T}_v$ are not parallel.
We now **divide up the domain** D^* into smaller rectangles;

$$R^*_{ij} = \{(u,v);\, u_i \le u \le u_i + \triangle u,\, v_j \le v \le v_j + \triangle v\}$$

The image of R^*_{ij} under the map $\mathbf{T}:(u,v) \to (x,y)$ is a small set R_{ij} approximately equal to the image under the linear map. The image under the linear map is a parallelogram with adjacent sides given by vectors $\mathbf{T}_u \triangle u$ and $\mathbf{T}_v \triangle v$. The area of the parallelogram is the magnitude of the crossproduct of the vectors in space:

$$|\mathbf{T}_u \triangle u \times \mathbf{T}_v \triangle v| = \left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} \right| \triangle u \triangle v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \triangle u \triangle v$$

and hence

$$\triangle A_{ij} = \text{Area}\,(R_{ij}) \sim \left| \frac{\partial(x,y)}{\partial(u,v)}(u_i,v_j) \right| \triangle u \triangle v$$

Now, with $(x_{ij},y_{ij}) = \big(x(u_i,v_j),y(u_i,v_j)\big) \in R_{ij}$ we have

$$\iint_{D^*} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} f\big(x(u_i,v_j),y(u_i,v_j)\big) \left| \frac{\partial(x,y)}{\partial(u,v)}(u_i,v_j) \right| \triangle u \triangle v$$

$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} f\big(x_{ij},y_{ij}\big) \triangle A_{ij} = \iint_D f(x,y)\,dxdy$$

since its a **Riemann sum** for the double integral.

Using the change of variable theorem in the plane.

Ex Let $D^* = \{(r,\theta);\ 0<r<1,\ 0\le\theta<2\pi\}$ and let $(x,y) = T(r,\theta) = (r\cos\theta, r\sin\theta)$. Then $D = T(D^*) = \{(x,y);\ 0<x^2+y^2\le 1\}$, since $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2\le 1$, so $T(D^*)\subset D$ and any point in D can be written as $(r\cos\theta, r\sin\theta)$. Moreover, $T(r,\theta) = T(r',\theta')$ implies that $(r,\theta) = (r',\theta')$, if both points are in D^*.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & r\sin\theta \\ \sin\theta & -r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r \neq 0.$$

In **polar coordinates** $x = x(r,\theta) = r\cos\theta,\ y = y(r,\theta) = r\sin\theta$ we get

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\big(\cos^2\theta + \sin^2\theta\big) = r$$

and hence $\displaystyle\iint_D f(x,y)\,dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\,r dr\,d\theta$

Ex. Find the volume of the unit ball $B = \{(x,y,z);\ x^2+y^2+z^2\le 1\}$.

Sol. $$\iiint_B 1\,dxdydz = \iint_{x^2+y^2\le 1}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\ dxdy = \int_{x^2+y^2\le 1} 2\sqrt{1-x^2-y^2}\,dxdy$$
$$= \int_0^{2\pi}\int_0^1 2\sqrt{1-r^2}\ r\,dr\,d\theta = -\int_0^{2\pi}\frac{2}{3}(1-r^2)^{3/2}\Big|_0^1 d\theta = \int_0^{2\pi}\frac{2}{3} = \frac{4\pi}{3}.$$

If $(x,y) = T(u,v)$ is an invertible mapping $(u,v) = T^{-1}(x,y)$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1}$$

i.e. the change of area going from (x,y) to (u,v) times the change going back is one.

Ex. Find $\displaystyle\iint_D xydxdy$, where $D = \{(x,y);\ 0\le 2x-y\le 2,\ \ 0\le y-x\le 1\}$.

Sol. Under the change of variables $u = 2x-y$ and $v = y-x$ the region becomes $D^* = \{(u,v)\ 0\le u\le 2,\ 0\le v\le 1\}$ and $x = u+v,\ y = 2u+v$. We have

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{\partial(u,v)}{\partial(x,y)}\right|^{-1} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}^{-1} = \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix}^{-1} = 1$$

so

$$\iint_D xy\,dxdy = \iint_{D^*}(u+v)(2u+v)\Big|\frac{\partial(x,y)}{\partial(u,v)}\Big|dudv = \int_0^2\int_0^1(u+v)(2u+v)\,dudv = \ldots = 9.$$