Lecture 13: 5.2 Double integrals over more general regions.
If D is a region of type I: $D = \{(x,y);\ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ then

$$\iint_D f(x,y)\,dA = \int_a^b \Big[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\Big]\,dx$$

In fact by the slice method

$$\iint_D f(x,y)\,dA = \int_a^b A(x)\,dx, \qquad \text{where} \quad A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy.$$

If D is a region of type II: $D = \{(x,y);\ c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$ then

$$\iint_D f(x,y)\,dA = \int_c^d \Big[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\Big]\,dy$$

Ex. Evaluate $\iint_D x\cos y\,dA$, where $D = \{(x,y);\ 0 \le x \le 1,\ 0 \le y \le x^2\}$. **Sol**

$$\iint_D x\cos y\,dA = \int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 x\sin y\Big|_{y=0}^{x^2} dx = \int_0^1 x\sin(x^2)\,dx$$
$$= \frac{\cos(x^2)}{2}\Big|_0^1 = \frac{\cos 1 - 1}{2}$$

5.3 Changing the order of integration.

Ex. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA, \quad D = \{(x,y);\ 0 \le y \le 1,\ 3y \le x \le 3\}$$

Here

$$D = \{(x,y);\ 0 \le y \le 1,\ 3y \le x \le 3\} = \{(x,y);\ 0 \le x \le 3,\ 0 \le y \le x/3\}$$

so

$$\iint_D e^{x^2} dA = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx = \int_0^3 e^{x^2} y\Big|_{y=0}^{x/3} dx = \int_0^3 e^{x^2}\frac{x}{3}\,dx = \frac{e^{x^2}}{6}\Big|_{x=0}^3 = \frac{e^9 - 1}{6}$$

Areas can be calculated using double integrals:

$$\iint_D 1\,dA = \text{Area}(D).$$

This is because the integral is the volume above D and below 1 which is $\text{Area}(D)\cdot 1$.

Lecture 13: Triple Integrals. We now want to define the integral of a function f over a rectangular box $B = \{(x, y, z); a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$. We divide B into smaller boxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ by dividing the interval $[a, b]$ into n subintervals of length $\triangle x = (b-a)/n$, $[c, d]$ into n subintervals of length $\triangle y$ and $[r, s]$ into n subintervals of length $\triangle z$. Then we form the Riemann sum

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\triangle V \tag{1}$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is a sample point in B_{ijk} and $\triangle V = \triangle x\, \triangle y\, \triangle z$. We then define the integral to be the limit of the Riemann sum:

$$\iiint_B f(x, y, z)\, dV = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\triangle V \tag{2}$$

Triple integrals do not have the same direct geometric interpretation as double integrals and volumes, because it is difficult to visualize four dimensional volumes. However, if $f(x, y, z)$ represent the density of mass per unit volume at a point (x, y, z) then the integral over B of f is the total mass of B.

As for the case of two variables we can write it as iterated integrals (Fubini):

$$\iiint_B f(x, y, z)\, dV = \int_r^s \Big(\int_c^d \Big(\int_a^b f(x, y, z)\, dx\Big)\, dy\Big)\, dz \tag{3}$$

or one can integrate the different variables in any other order.
Ex. Let $B = \{(x, y, z);\ 1 \le x \le 2,\ 0 \le y \le 1,\ 1 \le z \le 2\}$. Find $\iiint_B x^2yz\, dV$.
Sol. $\iiint_B x^2yz\, dV = \int_1^2 \int_1^2 \int_0^1 x^2yz\, dy\, dz\, dx = ...$

As for double integrals we define the integral of f over a more general bounded region E by finding a large box B containing E and integrating the function that is equal to f in E and 0 outside E over the lager box B.

We now restrict our attention to some special regions. Region of **type 1**:

$$E = \{(x, y, z);\ (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\} \tag{4}$$

where D is the projection of E onto the x-y plane. Then

$$\iiint_E f(x, y, z)\, dV = \iint_D \Big(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\Big) dA \tag{5}$$

In particular if D is a region of type I in the plane then

$$E = \{(x, y, z);\ a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\} \tag{6}$$

and

$$\iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dz \tag{7}$$

Note also that one can find the volume by using triple integrals

$$\text{Volume}(E) = \iiint_E 1\, dV \tag{8}$$

Ex. Let $E = \{(x, y, z);\ x \geq 0,\ y \geq 0,\ z \geq 0,\ x + 2y + 3z \leq 1\}$. Find the volume of E
Sol. $E = \{(x, y, z);\ 0 \leq x \leq 1,\ 0 \leq y \leq (1 - x)/2,\ 0 \leq z \leq (1 - x - 2y)/3\}$ so

$$V = \iiint_E 1 dV = \int_0^1 \int_0^{(1-x)/2} \int_0^{(1-x-2y)/3} dz\, dy\, dx = ...$$

Region of type 2: If $E = \{(x, y, z);\ (y, z) \in D,\ u_1(y, z) \leq x \leq u_2(y, z)\}$. then

$$\iiint_B f(x, y, z)\, dV = \iint_D \Big(\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \Big) dA$$

Region of type 3: If $E = \{(x, y, z);\ (x, z) \in D,\ u_1(x, z) \leq x \leq u_2(x, z)\}$ then

$$\iiint_B f(x, y, z)\, dV = \iint_D \Big(\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \Big) dA$$

Ex. Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$ where E is the region bounded by $y = x^2 + x^2$ and the plane $y = 4$.
Sol. $E = \{(x, y, z);\ (x, z) \in D,\ x^2 + z^2 \leq y \leq 4\}$, where $D = \{(x, z);\ x^2 + z^2 \leq 4\}$.

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_D \int_{x^2+z^2}^4 \sqrt{x^2 + z^2} dy\, dA = \iint_D \big(4 - (x^2 + z^2)\big) \sqrt{x^2 + z^2}\, dA$$

Introducing polar coordinates in the x-z plane: $x = r\cos\theta$, $z = r\sin\theta$, $dA = dxdz = rdrd\theta$ gives that the integral is: $\int_0^{2\pi} \int_0^2 (4 - r^2) r\, rdr\, d\theta = ... = \frac{128\pi}{15}$