**Lecture 12: 5.1-2 Double Integrals.**

**Areas and integrals** If $f \geq 0$ the integral is intuitively defined by the area:

$$\int_a^b f(x)\,dx = \text{Area below the graph } y = f(x) \text{ and above the } x \text{ axis, from } a \text{ to } b.$$

The area of a region is defined by filling in the region with many small rectangles and taking the limit of the total area of all the rectangles as they get smaller.

The integral of a function $f(x)$, of one variable, over the interval $[a,b]$ is defined as follows. We start by dividing the interval into $n$ subintervals $[x_{i-1}, x_i]$ of equal length $\triangle x = (b-a)/n$ and we choose sample points $x_i^*$ in these subintervals. Then we note that the area below the graph over a subinterval is approximately $f(x_i^*)\triangle x$. The are below the graph over the whole interval is therefore the sum $\sum_{i=1}^n f(x_i^*)\triangle x$. Taking the limit as $n \to \infty$, i.e. $\triangle x \to 0$, the approximation gets better, and we define

$$\int_a^b f\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\triangle x$$

**Def** A function of one variable is said to be **Integrable** if the limit above exist and is independent of the choice of points $x_i^* \in [x_{i-1}, x_i]$.
**Ex** The function on $[0,1]$ which is 1 on all the rational points (i.e. of the form $p/q$ where $p, q$ are integers) and 0 all irrationals is not integrable.
**Th** A continuous function on a closed interval is integrable.
**Pf** Suppose we have to choices $x_i^*, x_i^{**} \in [x_{i-1}, x_i]$. Since a continuous function on a closed interval is uniformly continuous; for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x_i^*) - f(x_i^{**})| \leq \varepsilon, \qquad \text{if} \qquad |x_i^* - x_i^{**}| \leq \delta.$$

Therefore if $n$ is so large that $(b-a)/n \leq \delta$ we have

$$\Big|\sum_{i=1}^n f(x_i^*)\triangle x - \sum_{i=1}^n f(x_i^{**})\triangle x\Big| = \Big|\sum_{i=1}^n \big(f(x_i^*) - f(x_i^{**})\big)\triangle x\Big| \leq \sum_{i=1}^n \big|f(x_i^*) - f(x_i^{**})\big|\triangle x$$

which is bounded by $\varepsilon \sum_{i=1}^n (b-a)/n = \varepsilon(b-a)$. Since $\varepsilon > 0$ was arbitrarily small we see that this can be made to to tend to 0 as $n \to \infty$.

**Volumes and double integrals** If $f \geq 0$ and $R = [a,b] \times [c,d]$ is a rectangle in the $x-y$ plane the double integral is intuitively defined by:

$$\iint_R f(x,y)\,dA = \text{Volume below the graph } z = f(x,y) \text{ and above the rectangle } R.$$

The volume of a solid is defined by filling in the solid with many small rectangular boxes and taking the limit of the total volume of all the boxes as they get smaller.

Suppose that $f(x,y)$ is a function of two variables defined on a rectangle $R = \{(x,y);\ a \leq x \leq b,\ c \leq y \leq d\}$ and suppose first that $f \geq 0$. Let $S$ be the solid that lies above $R$ and below the graph of $f$, i.e. $S = \{(x,y,z);\ 0 \leq z \leq f(x,y),\ (x,y) \in R\}$. We want to find the volume of $S$. The first step is to divide $R$ into subrectangles, by dividing $[a, b]$ into subintervals $[x_{i-1}, x_i]$ of equal length $\triangle x = (b-a)/n$ and dividing $[c, d]$ into subintervals $[y_{j-1}, y_j]$ of equal length $\triangle y = (d-c)/n$. We hence obtain subrectangles $R_{ij} = \{(x, y);\, x_{i-1} < x < x_i,\, y_{j-1} < y < y_j\}$ of equal area $\triangle A = \triangle x \triangle y$. Next we choose sample points $(x_{ij}^*, y_{ij}^*) \in R_{ij}$. Then we can approximate the volume $\triangle V_{ij}$ of the part of $S$ that lies above $R_{ij}$ by a small rectangular box with base $R_{ij}$ and height $f(x_{ij}^*, y_{ij}^*)$ so the volume $\triangle V_{ij}$ is $f(x_{ij}^*, y_{ij}^*)\triangle A$. The total volume of all the approximating rectangular boxes is

$$\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\triangle A$$

In the limit as $n \to \infty$ we expect this to converge to what we think of as the volume:

$$V = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\triangle A$$

In generally, we define the **double integral** of $f(x, y)$ over the rectangle $R$ to be

$$\iint_R f(x, y)\, dA = \lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\triangle A$$

**Def** A function of two variable is said to be **Integrable** if the limit above exist and is independent of the choice of points $(x_{ij}^*, y_{ij}^* \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]$.
**Th** A continuous function on a closed rectangle is integrable.
We have now defined double integrals but we also have to be able to calculate them:
**Slice method** Let $S$ be a solid lying between the planes $x = a$ and $x = b$. Let us slice $S$ by intersecting with planes $x = x_0$ and let $A(x_0)$ denoted the area of the cross section. Then we divide up the interval $[a, b]$ on the $x$-axis in $n$ small subintervals of thickness $\triangle x$ as in definition of integrals. The volume of the slab of thinkness $\triangle x$ is the approximately $A(x_i^*)\triangle x$ and the total volume of $S$ is approximately the sum $\sum_{i=1}^{n} A(x_i^*)\Delta x$. In the limit as $n \to \infty$ we obtain the integral:

$$\text{Volume of } S = \int_a^b A(x)\, dx$$

**Iterated Integrals** We now use the slice method to find the volume of the solid below the graph of $f(x, y)$ and above the rectangle $R = [a, b] \times [c, d]$ in the $x - y$ plane. The area $A(x_0)$ of the intersection of the solid with the plane $x = x_0$ is the area below the graph of $f(x_0, y)$ as a function of $y$ and above the $y$-interval $[c, d]$;

$$A(x) = \int_c^d f(x, y)\, dy$$

Hence

$$\text{Volume of } S = \int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx$$

This is called an **iterated integral**. Since we can change $x$ and $y$ we have shown:

**Th.(Fubini)** If $R=[a,b]\times[c,d]=\{(x,y);\ a\le x\le b,\ c\le y\le d\}$ then

$$\iint_R f(x,y)\,dA=\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx=\int_c^d\left[\int_a^b f(x,y)\,dx\right]dy.$$

**Ex.** Find the integral of $16-x^2-2y^2$ over the square $R=[0,2]\times[0,2]$. **Sol.**

$$\iint_R 16-x^2-2y^2\,dA=\int_0^2\int_0^2 16-x^2-2y^2\,dy\,dx=\int_0^2 16y-x^2y-\frac{2}{3}y^3\Big|_0^2\,dx$$

$$=\int_0^2 32-2x^2-\frac{16}{3}\,dx=32x-\frac{2}{3}x^3-\frac{16}{3}x\Big|_0^2=64-\frac{16}{3}-\frac{32}{3}=48$$