Lecture 10: 4.2: Maximum and minimum values.

Def. A function $f : \mathbf{R}^n \to \mathbf{R}$ has a **local maximum** at $\mathbf{a}$ if $f(\mathbf{x}) \leq f(\mathbf{a})$ when $\mathbf{x}$ is near $\mathbf{a}$. Similarly, f has a **local minimum** at $\mathbf{a}$ if $f(\mathbf{x}) \geq f(\mathbf{a})$ when $\mathbf{x}$ is near $\mathbf{a}$. If the inequalities hold for all $\mathbf{x}$ in the domain of f then it is also a **global** (or **absolute**) **maximum** respectively **global** (or **absolute**) **minimum**.

By a **local extremum** we mean a local maximum or minimum.

Th. If f has as a maximum or minimum at $\mathbf{a}$ then $\mathbf{D}f(\mathbf{a}) = 0$.

Pf. If f has a local max or min at $\mathbf{a}$ then for $\mathbf{h}$ the function $g(t) = f(\mathbf{a} + t\mathbf{h})$ has a local max or min at $t = 0$, so $g'(0) = 0$. By the chain rule we have

$$g'(t) = \frac{d}{dt} f(\mathbf{a} + t\mathbf{h}) = \mathbf{D}f(\mathbf{a} + t\mathbf{h})\mathbf{h} = \nabla f(\mathbf{a} + t\mathbf{h}) \cdot \mathbf{h}$$

Hence

$$0 = g'(0) = \nabla f(\mathbf{a}) \cdot \mathbf{h} = f_{x_1}(\mathbf{a})h_1 + \cdots + f_{x_n}(\mathbf{a})h_n$$

Since $\mathbf{h} = (h_1, \ldots, h_n)$ is arbitrary it follows that $f_{x_1}(\mathbf{a}) = \cdots = f_{x_n}(\mathbf{a}) = 0$.

Def. A point $\mathbf{a}$ is called a **critical point** of f if $\mathbf{D}f(\mathbf{a}) = 0$.

The geometric interpretation of the conclusion of the theorem in $\mathbf{R}^2$ is that the tangent plane to the surface $z = f(x, y)$ at (a, b) is horizontal.

By the above theorem a local maximum or local minimum has to be a critical point. However, not all critical points are local maximum or minimum. For functions of one variable take e.g. $f(x) = x^3$ at $x = 0$.

How do we know if a critical point is an extreme value? For functions of one variable, if $f'(a) = 0$ and $f''(a) > 0$ then it is a min and if $f''(a) < 0$ then it is a max and if $f''(a) = 0$ then it could be either or neither. What would the corresponding condition be in higher dimensions? Recall that by Taylor's formula if $\mathbf{D}f(\mathbf{a}) = 0$

$$f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) = \frac{1}{2}\mathbf{h}^T Hf(\mathbf{a})\mathbf{h} + R_2(\mathbf{a}, \mathbf{h}), \quad \text{where} \quad |R_2(\mathbf{a}, \mathbf{h})| \leq M\|\mathbf{h}\|^3$$

where $Hf = \{\partial_{x_i}\partial_{x_j} f\}_{i,j=1,\ldots,n}$ is the Hessian matrix of second derivatives of f. The term with the Hessian is a quadric form Q in $\mathbf{h}$, i.e. for some constants b_{ij}:

$$Q(\mathbf{h}) = \mathbf{h}^T B\mathbf{h} = \sum_{i,j=1,\ldots,n} b_{ij} h_i h_j$$

Def A quadratic form is called **positive definite** if $Q(\mathbf{h}) > 0$, for all $\mathbf{h} \neq \mathbf{0}$ and **negative definite** if $Q(\mathbf{h}) > 0$, for all $\mathbf{h} \neq \mathbf{0}$.

Th (Second derivative test) Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ has a critical point at $\mathbf{a}$.

(1) If the Hessian $Hf(\mathbf{a})$ is positive definite then f a local min at $\mathbf{a}$.

(2) If the Hessian $Hf(\mathbf{a})$ is negative definite then f a local max at $\mathbf{a}$.

(3) If $\det Hf(\mathbf{a}) \neq 0$ but neither (1) nor (2) hold then f has a saddle point at $\mathbf{a}$.

Th. (The second derivative test for functions of two variables) Suppose that (a, b) is a critical point and

$$D = f_{xx}(a, y) f_{yy}(a, b) - f_{xy}(a, b)^2.$$

If $D > 0$ and $f_{xx}(a, y) > 0$ then $f(a, b)$ is a local minimum.

If $D > 0$ and $f_{xx}(a, y) < 0$ then $f(a, b)$ is a local maximum.

If $D < 0$ then $f(a, b)$ is a not a local minimum or maximum.

If $D = 0$ then the test is inconclusive.

Proof To prove the first statement we have to show that the symmetric quadratic form $ax^2 + by^2 + (c+d)xy$ is positive definite if $a > 0$ and $ab - cd > 0$, where $c = d$. Completing the square we get

$$ax^2 + by^2 + 2cxy = a(x^2 + 2xyc/a + y^2b/a) = a\big((x + yc/a)^2 + y^2(b/a - c^2/a^2)\big) > 0$$

for all $(x, y) \neq (0, 0)$ if $a > 0$ and $ba - c^2 > 0$.

The case $D < 0$ is called a saddle point. To remember the formula:

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$$

Ex. Find the critical points of $f(x, y) = x^2 + y^2$. Are they extreme values?

Ex. Find the critical points of $f(x, y) = y^2 - x^2$ Are they extreme values?

Th.(Extreme value theorem) If f is continuous on a closed and bounded set D then f attains an absolute maximum and an absolute minimum in D.

Idea of proof If time permits.

To find the max and min in a closed and bounded domain:

1) Find the critical points in the domain.

2) Find the extreme values on the boundary.

Note that we do not need to use the second derivative test since by the extreme value theorem we know that there is a max and a min and any max or min either has to be a point on the boundary or a critical point in the interior.

In particular in one dimension. If we want to find the maximum of $f(x)$ over the interval $I = [a, b] = \{x;\ a \leq x \leq b\}$, then we first find all the critical points $f'(c_i) = 0$, $i = 1, ..., N$ and we check the value of f on these points and the boundary points a and b in order to find the largest and smallest value.

Ex. Find the max and min of $f(x, y) = x^2 + 2y^2$ over
(a) $D = \{(x, y);\ |x| \leq 1,\ |y| \leq 1\}$ and (b) $D = \{(x, y);\ x^2 + y^2 \leq 1\}$.

Sol. Critical points $f_x = 2x = 0$ and $f_y = 4y = 0$ is $(x, y) = (0, 0)$ and $f(0, 0) = 0$.
(a) Extreme value on the boundary. Divide boundary into the 4 parts. (1) $x = 1$ and $-1 \leq y \leq 1$: If $g(y) = f(1, y) = 1 + 2y^2$ then $g'(y) = 4y = 0$ if $y = 0$ and $g(0) = f(1, 0) = 1$. Endpoints of the interval: $g(1) = f(1, 1) = f(1, -1) = g(-1) = 3$. (2) $y = 1$ and $-1 \leq x \leq 1$. If $h(x) = f(x, 1) = x^2 + 1$ then $h'(x) = 2x = 0$ if $x = 0$ and $g(0) = f(0, 1) = 2$. Endpoints of the interval: $h(1) = f(1, 1) = f(-1, 1) = h(-1) = 3$. The other two parts of the boundary are the same so max is $f(\pm 1, \pm 1) = 3$ and min is $f(0, 0) = 0$.
(b) Extreme value on the boundary. Sol. 1: Parameterize boundary $(x, y) = (\cos t, \sin t)$. $g(t) = f(\cos t, \sin t) = \cos^2 t + 2\sin^2 t$. $g'(t) = -2\cos t \sin t + 4 \sin t \cos t = 2 \sin t \cos t = 0$ if $t = 0, \pi/2, \pi, 3\pi/2$. $g(0) = f(1, 0) = 1$, $g(\pi/2) = f(0, 1) = 2$, $g(\pi) = f(-1, 0) = 1$ and $g(3\pi/2) = f(0, -1) = 2$ so max is $f(0, \pm 1) = 2$ and min is $f(0, 0) = 0$. Sol2: Solve for y in $x^2 + y^2 = 1$ gives $y = \pm\sqrt{1 - x^2}$, $-1 \leq x \leq 1$. Substituting into $f(x, y)$ gives $h(x) = f(x, \pm\sqrt{1 - x^2}) = 2 - x^2$ and we want to maximize over $-1 \leq x \leq 1$. $h'(x) = -2x = 0$ if $x = 0$ and $h(0) = f(0, \pm 1) = 2$. Endpoints $h(\pm 1) = f(\pm 1, 0) = 1$.

Second derivative test in higher dimensions and idea of its proof
If time permits. See book.