Math 211 honors Multivariable Calculus

Lecture 1: Overview+Review of vector operations.
What is the multivariable calculus course about?
Curves in space $(x(t), y(t), z(t))$, e.g. path of a particle.
Vectors and vector operations, e.g. the dot and the cross product.
Equations of lines and planes. (using vectors)
The the velocity $\mathbf{v}(t) = (x'(t), y'(t), z'(t))$, tangent vector to the particle.
Functions of several variables $f(x, y, z)$, e.g. temperature at each point in space.
Derivatives of functions of several variables:
E.g. if f represent the temperature then the gradient vector $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ represents the direction in which the temperature increases the most.
Multiple integrals: Double integrals over domains in the plane and triple or volume integrals over domains in space. Polar coordinates.

Ch 1: Vectors and vector operations.
Ch 2: Derivatives of functions of several variables.
Ch 3: **Vector fields** $\mathbf{v}(x, y, z) = \big((v_1(x, y, z), v_2(x, y, z), v_3(x, y, z)\big)$, at each point in space we are given a vector, and the **divergence** and **curl** of them.
The gravitational field is a vector field; at each point in space we are given a vector. The vector field of a continuum of fluid particles is a vector field. At each point in space we are given the velocity of the particles at that point.

Each component of a vector field can be differentiated in each direction. The divergence and curl are special derivatives with physical meaning.
Ch 4: Mx/min in several variables.
Ch 5 Double and Triple Integrals and the **change of variable theorem** in multiple integrals.
Ch 6 Line Integrals and Green's theorem. Generalization of the fundamental theorem of calculus $\int_a^b f'(x)\, dx = f(b) - f(a)$ to several variables: E.g. $F = (F_1, F_2, 0)$ a vector function and D a domain in the $x - y$ plane with boundary C then $\displaystyle\int_C F_1 dx + F_2 dy = \iint_D (\partial F_2/\partial x - \partial F_1/\partial y)\, dx$.
Ch 7 **Surface area and surface Integrals flow through surface** and Stoke's and Gauss's theorems.
Ch 8: **Differential forms.**

Surface integrals is the hardest part. Thinking geometrically and physically helps understanding concepts but is not needed to do problems

Section 1.1 Vectors.

Recall that a **point** in space can be represented by an ordered triplet (x, y, z) of real numbers called the **Cartesian or rectangular coordinates**, that measure the lengths of the projections on the three coordinate axis.

A **vector a** is a directed line segment or arrow; it has a length $\|\mathbf{a}\|$ and a direction. Its components (a_1, a_2, a_3) are the coordinates of the endpoint if it starts at the origin. In the notes vectors will be denoted by boldface letters **a** and in the lectures by $\overrightarrow{a}$.

Addition of vectors is geometrically defined by the triangle law and algebraically by $(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.
Scalar multiplication $\alpha\,\mathbf{a}$ is a vector in the same direction as **a** with length $|\alpha|\,\|\mathbf{a}\|$ and its algebraically given by $\alpha(a_1, a_2, a_3) = (\alpha a_1, \alpha a_2, \alpha a_3)$.

The **length** of the vector is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, by the Pythagorean theorem.
A **unit vector** (i.e. of length one) in the direction of the vector **a** is given by $\dfrac{\mathbf{a}}{\|\mathbf{a}\|}$.

Section 1.2: Basis and what we can do with vectors?

Standard basis: If $\mathbf{i}=(1, 0, 0)$, $\mathbf{j}=(0, 1, 0)$ and $\mathbf{k}=(0, 0, 1)$ then
$\mathbf{a}= (a_1, a_2, a_3) = a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are points in space then the vector represented by the directed line segment $\overrightarrow{P_1P_2}$ is $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_2)\mathbf{k}$.
Ex Find the parametric equations of a line passing through $(0, 1, 2)$ and $(1, 2, 4)$.
Sol The vector $\mathbf{v} = (1, 2, 4) - (0, 1, 2) = (1, 1, 2)$ is parallel to the line and so is any multiple $t\mathbf{v}$. The points on the line are therefore given by
$(x, y, z) = (0, 1, 2) + t(1, 1, 2) = (t, 1 + t, 2 + 2t)$.

Section 1.3:. Inner product (or scalar or dot product) of two vectors is

$$\mathbf{a} \cdot \mathbf{b} = (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3 \tag{1}$$

The geometric interpretation is $\|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$, where θ is the angle between **a** and **b**. Note that $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ and that $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if **a** and **b** are perpendicular. Note that $\|\mathbf{b}\| \cos\theta$ is the component of **b** in the direction of **a**.
The orthogonal projection **p** of **b** on **a** is given by $\mathbf{p} = \|\mathbf{b}\| \cos\theta \dfrac{\mathbf{a}}{\|\mathbf{a}\|} = \dfrac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$.

Ex. Decompose $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ into a vector $\mathbf{b}_{\|}$ parallel to $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and a vector $\mathbf{b}_{\perp}$ perpendicular to **a**.
Sol. $\mathbf{a} \cdot \mathbf{b} = 2 - 2 + 4 = 4$, $\|\mathbf{a}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ so $\mathbf{b}_{\|} = (\mathbf{a}\cdot\mathbf{b})\ \mathbf{a}/\|\mathbf{a}\|^2 = 4(\mathbf{i} + \mathbf{j} + \mathbf{k})/3$ and $\mathbf{b}_{\perp} = \mathbf{b} - \mathbf{b}_{\|} = (2 - 4/3)\mathbf{i} - (1 + 4/3)\mathbf{j} + (4 - 4/3)\mathbf{k}$.

Section 1.4. The **vector or cross product** is the vector

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} + (a_3 b_1 - a_1 b_3)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.$$

The geometric interpretation is $\|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta\,\mathbf{n}$, where $\mathbf{n}$ is a unit vector, $\|\mathbf{n}\| = 1$, that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and pointing in the direction so that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{n}$ form a positively oriented system.
Note that $\mathbf{a} \times \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel.

To remember the definition of vector product we introduce so called **determinants**.
A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{2}$$

Its magnitude is the **area** of the parallelogram with vectors (a, b) and (c, d) as edges.
A **determinant of order 3** is defined by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{3}$$

Its magnitude is the **volume** of the parallelepiped with vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ and $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ as edges.

The cross product (2) is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} \tag{4}$$

Because of the similarity with (3), to remember this we symbolically write

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{5}$$