23. Let S be the surface of region W. Show that

$$\iint_S \mathbf{r} \cdot \mathbf{n}\, dS = 3 \text{ volume } (W).$$

Explain this geometrically.

8.5 Some Differential Equations of Mechanics and Technology

Isaac Newton reputedly said, "All in nature reduces to differential equations." This point of view was paraphrased by Max Planck (see the Historical Note in Section 3.3): "... Present day physics, as far as it is theoretically organized, is completely governed by a system of space–time differential equations."

In this section, we apply the central theorems of vector analysis to the derivation of the differential equations governing heat transfer, electromagnetism, and the motion of some fluids.

Keep in mind the importance of these problems in modern technology. For example, a good understanding of fluids and the ability to do computations to solve their governing equations is at the heart of how one builds a modern airplane or designs a submarine. For instance, the flow of air (the fluid in this case) over the wings of an aircraft is very subtle, even though the governing equations are relatively simple. We shall derive a slightly idealized form of these equations in this section. Likewise, the equations of electromagnetism, as we will discuss in the following paragraphs, is central to the communications industry; wireless, television, and much of the operation of modern electronic devices, including computers, depends on these and related fundamental equations.

Conservation Laws

As preparation for deriving the equations of a fluid, let us first discuss an important equation that is referred to as a ***conservation*** equation. For fluids, it expresses the conservation of mass; for electromagnetic theory, it expresses the conservation of charge. We shall apply these ideas to the equation for heat conduction and to electromagnetism.

Let $\mathbf{V}(t, x, y, z)$ be a C^1 vector field on $\mathbb{R}^3$ for each t and let $\rho(t, x, y, z)$ be a C^1 real-valued function. By the ***law of conservation of mass*** for $\mathbf{V}$ and ρ, we mean that the condition

$$\frac{d}{dt}\iiint_W \rho\, dV = -\iint_{\partial W} \mathbf{J} \cdot \mathbf{n}\, dS$$

holds for all regions W in $\mathbb{R}^3$, where $\mathbf{J} = \rho\mathbf{V}$ (see Figure 8.5.1).

If we think of ρ as a mass density (ρ could also be a charge density)—that is, the mass per unit volume—and of $\mathbf{V}$ as the velocity field of a fluid, the condition simply says that the rate of change of total mass in W equals the rate at which mass flows *into* W. Recall that $\iint_{\partial W} \mathbf{J} \cdot \mathbf{n}\, dS$ is called the *flux* of $\mathbf{J}$. We need the following result.

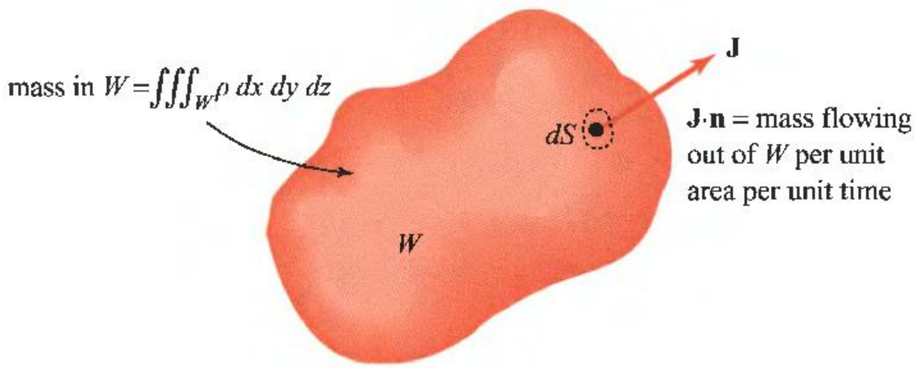

Figure 8.5.1 The rate of change of mass in W equals the rate at which mass crosses ∂W.

THEOREM 11 For $\mathbf{V}$ and ρ (a smooth vector field and a scalar field on $\mathbb{R}^3$), the law of conservation of mass for $\mathbf{V}$ and ρ is equivalent to the condition

$$\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \tag{1}$$

That is,

$$\rho \operatorname{div} \mathbf{V} + \mathbf{V} \cdot \nabla \rho + \frac{\partial \rho}{\partial t} = 0. \tag{1$'$}$$

Here, div $\mathbf{J}$ means that we compute div $\mathbf{J}$ for t held fixed, and $\partial\rho/\partial t$ means we differentiate ρ with respect to t for x, y, z fixed.

PROOF First, observe that by differentiating under the integral, we get

$$\frac{d}{dt}\iiint_W \rho\, dx\, dy\, dz = \iiint_W \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

and also

$$\iint_{\partial W} \mathbf{J}\cdot\mathbf{n}\, dS = \iiint_W \operatorname{div} \mathbf{J}\, d\mathbf{V}$$

by the divergence theorem. Thus, conservation of mass is equivalent to the condition

$$\iiint_W \left(\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t}\right) dx\, dy\, dz = 0.$$

Because this is to hold for all regions W, it is equivalent to $\operatorname{div} \mathbf{J} + \partial\rho/\partial t = 0$. ■

The equation $\operatorname{div} \mathbf{J} + \partial\rho/\partial t = 0$ is called the ***equation of continuity***. An interesting remark is that using the change of variables formula, the law of conservation of mass may be shown to be equivalent to the condition

$$\frac{d}{dt}\iiint_{W_t} \rho\, dV = 0,$$

where W_t is the image of W obtained by moving each point in W along flow lines of $\mathbf{V}$ for time t. This result is a special case of the *transport theorem* that we discuss next.

The Transport Theorem

The transport theorem is an interesting application of the divergence theorem that will be needed in our derivation of the equations of a fluid.

THEOREM 12 Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ and denote the flow line of $\mathbf{F}$ starting at $\mathbf{x}$ after time t by $\phi(\mathbf{x}, t)$. (See the Internet supplement to Section 4.4 for more information.) Let $J(\mathbf{x}, t)$ be the Jacobian of the map $\phi_t\colon \mathbf{x} \mapsto \phi(\mathbf{x}, t)$ for t fixed. Then

$$\frac{\partial}{\partial t} J(\mathbf{x}, t) = [\operatorname{div} \mathbf{F}(\phi(\mathbf{x}, t))] J(\mathbf{x}, t).$$

For a given function $f(x, y, z, t)$ and a region $W \subset \mathbb{R}^3$, the ***transport equation*** holds:

$$\frac{d}{dt}\iiint_{W_t} f(x, y, z, t)\, dx\, dy\, dz = \iiint_{W_t} \left(\frac{Df}{Dt} + f \operatorname{div} \mathbf{F}\right) dx\, dy\, dz,$$

where $W_t = \phi_t(W)$, which is the region moving with the flow, and where

$$\frac{Df}{Dt} = \partial f/\partial t + \nabla f \cdot \mathbf{F}$$

is the ***material derivative***.

Taking $f = 1$, Theorem 12 implies that the following assertions are equivalent (which justifies the use of the term *incompressible*):

1. $\operatorname{div} \mathbf{F} = 0$
2. volume (W_t) = volume (W)
3. $J(\mathbf{x}, t) = 1$

Let ϕ, J, $\mathbf{F}$, f be as just defined. There is also a vector form of the transport theorem, namely,

$$\frac{d}{dt}\iiint_{W_t} (f\mathbf{F})\, dx\, dy\, dz$$

$$= \iiint_{W_t} \left[\frac{\partial}{\partial t}(f\,\mathbf{F}) + \mathbf{F} \cdot \nabla(f\,\mathbf{F}) + (f\,\mathbf{F}) \operatorname{div} \mathbf{F}\right] dx\, dy\, dz,$$

where $\mathbf{F}\cdot\nabla(f\mathbf{F})$ denotes the 3×3 derivative matrix $\mathbf{D}(f\mathbf{F})$ operating on the column vector $\mathbf{F}$; in Cartesian coordinates, $\mathbf{F}\cdot\nabla\mathbf{G}$ is the vector whose ith component is

$$\sum_{j=1}^{3} F_j \frac{\partial G^i}{\partial x_j} = F_1\frac{\partial G^i}{\partial x} + F_2\frac{\partial G^i}{\partial y} + F_3\frac{\partial G^i}{\partial z}.$$

We shall leave the proofs of these results, which are extensions of the arguments used to prove Theorem 11, to the reader (see the exercises).

Derivation of Euler's Equation of a Perfect Fluid

The continuity equation is not sufficient to completely determine the motion of a fluid—we need other conditions.

The fluids that the continuity equation governs can be compressible. If $\operatorname{div}\mathbf{V}=0$ (incompressible case) and ρ is constant, equation (1′) follows automatically. But in general, even for incompressible fluids, the equation is not automatic, because ρ can depend on (x, y, z) and t. Thus, even if the equation $\operatorname{div}\mathbf{V}=0$ holds, $\operatorname{div}(\rho\mathbf{V})\neq 0$ may still be true.

Here we discuss Euler's equation for a perfect fluid. Consider a nonviscous fluid moving in space with a velocity field $\mathbf{V}$. When we say that the fluid is *perfect*, we mean that if W is any portion of the fluid, forces of pressure act on the boundary of W along its normal. We assume that the force per unit area acting on ∂W is $-p\mathbf{n}$, where $p(x, y, z, t)$ is some function called the ***pressure*** (see Figure 8.5.2). Thus, the total pressure force acting on W is

$$\mathbf{F}_{\partial W} = \text{force} = -\iint_{\partial W} p\mathbf{n}\, dS.$$

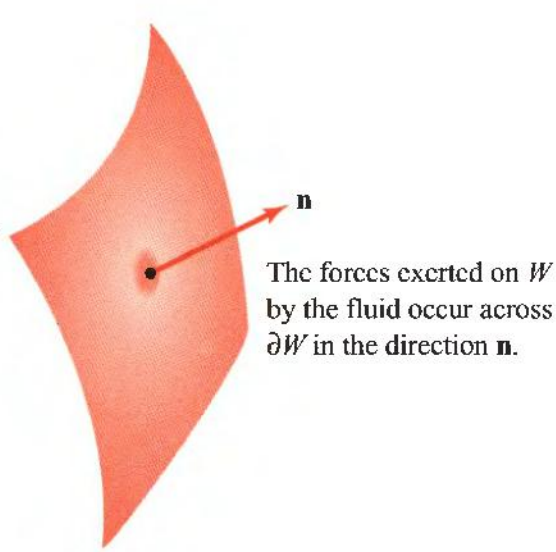

Figure 8.5.2 The force acting on ∂W per unit area is $-p\mathbf{n}$.

This is a *vector* quantity; the ith component of $\mathbf{F}_{\partial W}$ is the integral of the ith component of $p\mathbf{n}$ over the surface ∂W (this is therefore the surface integral of a real-valued function). If $\mathbf{e}$ is any fixed vector in space, we have

$$\mathbf{F}_{\partial W} \cdot \mathbf{e} = -\iint_{\partial W} p\mathbf{e} \cdot \mathbf{n}\, dS,$$

which is the integral of a scalar over ∂W. By the divergence theorem and identity (7) in the table of vector identities (Section 4.4), we get

$$\mathbf{E} \cdot \mathbf{F}_{\partial W} = -\iiint_W \operatorname{div}(p\mathbf{E})\, dx\, dy\, dz = -\iiint_W (\operatorname{grad} p) \cdot \mathbf{E}\, dx\, dy\, dz,$$

so that

$$\mathbf{F}_{\partial W} = -\iiint_W \nabla p\, dx\, dy\, dz.$$

Now we apply *Newton's second law* to a moving region W_t. As in the transport theorem, $W_t = \phi_t(W)$, where $\phi_t(\mathbf{x}) = \phi(\mathbf{x}, t)$ denotes the flow of $\mathbf{V}$. The rate of change of momentum of the fluid in W_t equals the force acting on it:

$$\frac{d}{dt}\iiint_{W_t} p\mathbf{V}\, dx\, dy\, dz = \mathbf{F}_{\partial W_t} = \iiint_{W_t} \nabla p\, dx\, dy\, dz.$$

We apply the vector form of the transport theorem to the left-hand side to get

$$\iiint_{W_t} \left[\frac{\partial}{\partial t}(\rho\mathbf{V}) + \mathbf{V} \cdot \nabla(\rho\mathbf{V}) + p\mathbf{V} \operatorname{div} \mathbf{V}\right] dx\, dy\, dz = -\iiint_{W_t} \nabla p\, dx\, dy\, dz.$$

Because W_t is arbitrary, this is equivalent to

$$\frac{\partial}{\partial t}(\rho\mathbf{V}) + \mathbf{V} \cdot \nabla(\rho\mathbf{V}) + \rho\mathbf{V} \operatorname{div} \mathbf{V} = -\nabla p.$$

Simplification using the equation of continuity, namely, formula (1′), gives

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla\mathbf{V}\right) = -\nabla p. \tag{2}$$

This is ***Euler's equation for a perfect fluid***. For compressible fluids, p is a given function of ρ (for instance, for many gases, $p = A\rho^{\gamma}$ for constants A and γ). On the other hand, if the fluid is incompressible, ρ is to be determined from the condition $\operatorname{div} \mathbf{V} = 0$. Equations (1) and (2) then govern the motion of the fluid.

Historical Note

The equations describing the motion of a fluid were first derived by Leonhard Euler in 1755, in a paper entitled "General Principles of the Motion of Fluids." Euler did basic work in mechanics as well as voluminous work in pure mathematics, a small part of which has already been discussed in this book; he essentially began the subject of analytical mechanics (as opposed to the Euclidean geometric methods used by Newton). He is responsible for the equations of a rigid body (equations that apply, for example, to a tumbling satellite) and the formulation of many basic equations of mechanics in terms of variational principles; that is, by the methods of maxima and minima of real-valued functions. Euler wrote the first comprehensive textbook on calculus and contributed to virtually all branches of mathematics. He wrote several books and hundreds of research papers even after he became totally blind, and he was working on a new treatise on fluid mechanics at the time of his death in 1783. Euler's equations for a fluid were eventually modified by Navier and Stokes to include viscous effects; the resulting Navier–Stokes equations are described in virtually every textbook on fluid mechanics.[7] Stokes is, of course, also responsible for developing Stokes' theorem, one of the main results discussed in this text!

Conservation of Energy and the Derivation of the Heat Equation

If $T(t, x, y, z)$ (a C^2 function) denotes the temperature in a body at time t, then ∇T represents the temperature gradient: Heat "flows" with the vector field $-\nabla T = \mathbf{F}$. Note that ∇T points in the direction of *increasing* T. Because heat flows from hot to cold, we have inserted a minus sign to reflect this physically observable fact. The energy density, that is, the energy per unit volume, is $c\rho_0 T$, where c is a constant (called the specific heat) and ρ_0 is the mass density, assumed constant. (We accept these assertions from elementary physics.) The ***energy flux vector*** is defined to be $\mathbf{J} = k\mathbf{F}$, where k is a constant called the ***conductivity***.

One now makes the hypothesis that energy is conserved. This means that $\mathbf{J}$ and $\rho = c\rho_0 T$ should obey the law of conservation of mass, with ρ playing the role of "mass" (note that it is *energy density*, not mass); that is,

$$\frac{d}{dt}\iiint_W \rho\, dV = -\iint_{\partial W} \mathbf{J}\cdot\mathbf{n}\, dS.$$

By Theorem 11, this assertion is equivalent to

$$\operatorname{div}\mathbf{J} + \frac{\partial\rho}{\partial t} = 0.$$

[7]The Clay Foundation has offered a prize of \$1 million to anyone who shows that for the incompressible Navier–Stokes equations, smooth data at $t = 0$ lead to smooth solutions for all $t > 0$.

But

$$\operatorname{div} \mathbf{J} = \operatorname{div}(-k\nabla T) = -k\nabla^2 T.$$

(Recall that $\nabla^2 T = \partial^2 T/\partial x^2 + \partial^2 T/\partial y^2 + \partial^2 T/\partial z^2$ and ∇^2 is the Laplace operator.) Continuing, we have

$$\frac{\partial \rho}{\partial t} = \frac{\partial(c\rho_0 T)}{\partial t} = c\rho_0 \frac{\partial T}{\partial t}.$$

Thus, the equation $\operatorname{div} \mathbf{J} + \partial\rho/\partial t = 0$ becomes

$$\frac{\partial T}{\partial t} = \frac{k}{c\rho_0}\nabla^2 T = \kappa\nabla^2 T, \tag{3}$$

where $\kappa = k/c\rho_0$ is called the ***diffusivity***. Equation (3) is the important ***heat equation***.

Just as equations (1) and (2) govern the flow of an ideal fluid, equation (3) governs the conduction of heat in the following sense. If $T(0, x, y, z)$ is a given initial temperature distribution, then a unique $T(t, x, y, z)$ is determined that satisfies equation (3). In other words, the initial condition at $t = 0$ gives the result for $t > 0$. Notice that if T does not change with time (the steady-state case), then we must have $\nabla^2 T = 0$ (Laplace's equation).

Maxwell's Equations and the Prediction of Radio Waves: The Communication Revolution Begins

We now return to *Maxwell's equations*, which govern the propagation of electromagnetic fields. The form of these equations depends on the physical units one is employing, and changing units introduces factors like 4π and the velocity of light. We shall choose the system in which Maxwell's equations are simplest.

Let $\mathbf{E}$ and $\mathbf{H}$ be C^1 functions of (t, x, y, z) that are vector fields for each t. They satisfy (by definition) ***Maxwell's equation with charge density*** $\rho(t, x, y, z)$ and ***current density*** $\mathbf{J}(t, x, y, z)$ when the following conditions hold:

$$\nabla \cdot \mathbf{E} = \rho \text{ (Gauss' law)}, \tag{4}$$

$$\nabla \cdot \mathbf{H} = 0 \text{ (no negative sources)}, \tag{5}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = \mathbf{0} \text{ (Faraday's law)}, \tag{6}$$

and

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = J \text{ (Ampère's law)}, \tag{7}$$

Of these laws, equations (4) and (6) were described in integral form in Sections 8.2 and 8.4; historically, they arose in these forms as physically observed laws. Ampère's law was mentioned for a special case in Section 7.2, Example 12.

Physically, one interprets $\mathbf{E}$ as the ***electric field*** and $\mathbf{H}$ as the ***magnetic field***. According to the preceding equations, as time t progresses, these fields interact with each other, and with any charges and currents that are present. For example, the propagation of electromagnetic waves (TV signals, radio waves, light from the sun, etc.) in a vacuum is governed by these equations with $\mathbf{J} = 0$ and $\rho = 0$.

Because $\nabla \cdot \mathbf{H} = 0$, we can apply Theorem 8 (from Section 8.3) to conclude that $\mathbf{H} = \nabla \times \mathbf{A}$ for some vector field $\mathbf{A}$. (We are assuming that $\mathbf{H}$ is defined on all of $\mathbb{R}^3$ for each time t.) The vector field $\mathbf{A}$ is not unique, and we can use $\mathbf{A}' = \mathbf{A} + \nabla f$ equally well for any function $f(t, x, y, z)$, because $\nabla \times \nabla f = 0$. (This freedom in the choice of $\mathbf{A}$ is called *gauge freedom*.) For any such choice of $\mathbf{A}$, we have, by equation (6),

$$\begin{aligned}\mathbf{0} = \nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} &= \nabla \times \mathbf{E} + \frac{\partial}{\partial t} \nabla \times \mathbf{A} \\ &= \nabla \times \mathbf{E} \times \nabla \times \frac{\partial \mathbf{A}}{\partial t} \\ &= \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right).\end{aligned}$$

Applying Theorem 7 (from Section 8.3), there is a real-valued function ϕ on $\mathbb{R}^3$ such that

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi.$$

Substituting this equation and $\mathbf{H} = \nabla \times \mathbf{A}$ into equation (7), and using the vector identity (whose proof we leave as an exercise)

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A},$$

we get

$$\begin{aligned}\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times (\nabla \times \mathbf{A}) - \frac{\partial}{\partial t} \left(-\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \right) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{\partial^2 \mathbf{A}}{\partial t^2} + \frac{\partial}{\partial t}(\nabla \phi).\end{aligned}$$

Thus,

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mathbf{J} + \nabla(\nabla \cdot \mathbf{A}) + \frac{\partial}{\partial t}(\nabla \phi).$$

That is,

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mathbf{J} + \nabla \left(\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} \right). \tag{8}$$

Again using the equation $\mathbf{E} + \partial \mathbf{A}/\partial t = -\nabla \phi$ and the equation $\nabla \cdot \mathbf{E} = \rho$, we obtain

$$\rho = \nabla \cdot \mathbf{E} = \nabla \cdot \left(-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right) = -\nabla^2 \phi - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t}.$$

That is,

$$\nabla^2 \phi = -\rho - \frac{\partial (\nabla \cdot \mathbf{A})}{\partial t}. \tag{9}$$

Now let us exploit the freedom in our choice of $\mathbf{A}$. We impose the "condition"

$$\nabla \cdot \mathbf{A} + \frac{\partial \phi}{\partial t} = 0. \tag{10}$$

We must be sure we can do this. Supposing we have a given $\mathbf{A}_0$ and a corresponding ϕ_0, can we choose a new $\mathbf{A} = \mathbf{A}_0 + \nabla f$ and then a new ϕ such that $\nabla \cdot \mathbf{A} + \partial \phi/\partial t = 0$? With this new $\mathbf{A}$, the new ϕ is $\phi_0 - \partial f/\partial t$; we leave verification as an exercise for the reader. Condition (10) on f then becomes

$$0 = \nabla \cdot (\mathbf{A}_0 + \nabla f) = \frac{\partial(\phi_0 - \partial f/\partial t)}{\partial t} = \nabla \cdot \mathbf{A}_0 + \nabla^2 f + \frac{\partial \phi_0}{\partial t} - \frac{\partial^2 f}{\partial t^2}$$

or

$$\nabla^2 f - \frac{\partial^2 f}{\partial t^2} = -\left(\nabla \cdot \mathbf{A}_0 + \frac{\partial \phi_0}{\partial t} \right). \tag{11}$$

Thus, to be able to choose $\mathbf{A}$ and ϕ satisfying $\nabla \cdot \mathbf{A} + \partial \phi/\partial t = 0$, we must be able to solve equation (11) for f. One can indeed do this under general conditions, although we do not prove it here. Equation (11) is called the ***inhomogeneous wave equation***.

If we accept that $\mathbf{A}$ and ϕ can be chosen to satisfy $\nabla \cdot \mathbf{A} + \partial \phi/\partial t = 0$, then equations (8) and (9) for $\mathbf{A}$ and ϕ become

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mathbf{J}; \tag{8$'$}$$

$$\nabla^2 \phi - \frac{\partial^2 \phi}{\partial t^2} = -\rho. \tag{9$'$}$$

Conversely, if $\mathbf{A}$ and ϕ satisfy the equations $\nabla \cdot \mathbf{A} + \partial\phi/\partial t = 0$, $\nabla^2\phi - \partial^2\phi/\partial t^2 = -\rho$, and $\nabla^2\mathbf{A} - \partial^2\mathbf{A}/\partial t^2 = -\mathbf{J}$, then $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and $\mathbf{H} = \nabla \times \mathbf{A}$ satisfy Maxwell's equations. *This procedure then "reduces" Maxwell's equations to a study of the wave equation.*[8]

Since the eighteenth century, solutions to the wave equation have been well studied (one learns these in most courses on differential equations). To indicate the wavelike nature of the solutions, for example, observe that for any function f,

$$\phi(t, x, y, z) = f(x - t)$$

solves the wave equation $\nabla^2\phi - (\partial^2\phi/\partial t^2) = 0$. This solution just propagates the graph of f like a wave; thus, one might conjecture that solutions of Maxwell's equations are wavelike in nature. Historically, all of this was Maxwell's great achievement, and it soon led to Hertz's discovery of radio waves. To quote from the Feynman *Lectures on Physics* (Vol. II):

> From a long view of the history of mankind—seen from, say, ten thousand years from now—there can be little doubt that the most significant event of the nineteenth century will be judged as Maxwell's discovery of the laws of electrodynamics. The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade.

Mathematics again shows its uncanny ability not only to *describe* but to *predict* natural phenomena.

There are other techniques (called Green's function methods) for dealing with the basic equations of mechanics and mathematical physics that also rely on vector calculus. Some of these methods are discussed in the Internet supplement for this book.

EXERCISES

1. Use a direct argument (or the proof of Theorem 1 in the Internet supplement to Section 4.4) to show that

$$\frac{\partial}{\partial t} J(\mathbf{x}, t) = [\operatorname{div} \mathbf{F}(\phi(\mathbf{x}, t))] J(\mathbf{x}, t).$$

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[8] There are variations on this procedure. For further details, see, for example, *Differential Equations of Applied Mathematics*, by G. F. D. Duff and D. Naylor, Wiley, New York, 1966, or books on electromagnetic theory, such as *Classical Electrodynamics*, by J. D. Jackson, Wiley, New York, 1962.

2. Using the change of variables theorem and Exercise 1, show that if $f(x, y, z, t)$ is a given function and $W \subset \mathbb{R}^3$ is any region, then the ***transport equation*** holds:

$$\frac{d}{dt}\iiint_{W_t} f(x, y, z, t)\, dx\, dy\, dz = \iiint_{W_t}\left(\frac{Df}{Dt} + f\,\mathrm{div}\,\mathbf{F}\right) dx\, dy\, dz$$

where $W_t = \phi_t(W)$, which is the region moving with the flow, and where $Df/Dt = \partial f/\partial t + \nabla f \cdot \mathbf{F}$ is the material derivative.

3. Use the transport equation to show that

$$\frac{d}{dt}\iiint_{W_t} \rho\, dx\, dy\, dz = 0$$

is equivalent to the law of conservation of mass.

4. Using Exercise 3 and the change of variables theorem, show that $\rho(\mathbf{x}, t)$ can be expressed in terms of the Jacobian $J(\mathbf{x}, t)$ of the flow map $\phi(\mathbf{x}, t)$ and $\rho(\mathbf{x}, 0)$ by the equation

$$\rho(\mathbf{x}, t)J(\mathbf{x}, t) = \rho(\mathbf{x}, 0).$$

What can you conclude from this for incompressible flow?

5. Prove the vector form of the transport theorem, namely,

$$\frac{d}{dt}\iiint_{W_t} (f\mathbf{F})\, dx\, dy\, dz = \iiint_{W_t}\left[\frac{\partial}{\partial t}(f\mathbf{F}) + \mathbf{F}\cdot\nabla(f\mathbf{F}) + (f\mathbf{F})\,\mathrm{div}\,\mathbf{F}\right] dx\, dy\, dz,$$

where $\mathbf{F}\cdot\nabla(f\mathbf{F})$ denotes the 3×3 derivative matrix $\mathbf{D}\,(f\mathbf{F})$ operating on the column vector $\mathbf{F}$; in Cartesian coordinates, $\mathbf{F}\cdot\nabla\mathbf{G}$ is the vector whose ith component is

$$\sum_{j=1}^{3} F_j \frac{\partial G^i}{\partial x_j} = F_1\frac{\partial G^i}{\partial x} + F_2\frac{\partial G^i}{\partial y} + F_3\frac{\partial G^i}{\partial z}.$$

6. Let $\mathbf{V}$ be a vector field with flow $\phi(\mathbf{x}, t)$ and let $\mathbf{V}$ and ρ satisfy the law of conservation of mass. Let W_t be the region transported with the flow. Prove the following version of the transport theorem:

$$\frac{d}{dt}\iiint_{W_t} \rho f\, dx\, dy\, dz = \iiint_{W_t} \rho\frac{Df}{Dt}\, dx\, dy\, dz.$$

7. (*Bernoulli's law*) (a) Let $\mathbf{V}$, ρ satisfy the law of conservation of mass and equation (2) (Euler's equation for a perfect fluid). Suppose $\mathbf{V}$ is irrotational and hence that $\mathbf{V} = \nabla\phi$ for a function ϕ. Show that if C is a path connecting two points P_1 and P_2, then

$$\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\|\mathbf{V}\|^2\right)\Bigg|_{\mathrm{P}_1}^{\mathrm{P}_2} + \int_C \frac{dp}{\rho} = 0.$$

[HINT: You may need the vector identity, $(\mathbf{V}\cdot\nabla)\mathbf{V} = \frac{1}{2}\nabla(\|\mathbf{V}\|^2) + (\nabla \times \mathbf{V}) \times \mathbf{V}$.]

(b) If in part (a), **V** is stationary—that is, $\partial \mathbf{V}/\partial t = 0$—and ρ is constant, show that

$$\frac{1}{2}\|\mathbf{V}\|^2 + \frac{p}{\rho}$$

is constant in space. Deduce that, in this situation, *higher pressure is associated with lower fluid speed.*

8. Using Exercise 7, show that if ϕ satisfies Laplace's equation $\nabla^2\phi = 0$, then $\mathbf{V} = \nabla\phi$ is a stationary solution to Euler's equation for a perfect *incompressible* fluid with constant density.

9. Verify that Maxwell's equations imply the equation of continuity for **J** and ρ.

10. For a steady-state charge distribution and divergence-free current distribution, the electric and magnetic fields $\mathbf{E}(x, y, z)$ and $\mathbf{H}(x, y, z)$ satisfy

$$\nabla \times \mathbf{E} = \mathbf{0}, \qquad \nabla\cdot\mathbf{H} = 0, \qquad \nabla\cdot\mathbf{J} = 0, \qquad \nabla\cdot\mathbf{E} = \rho, \qquad \text{and} \qquad \nabla \times \mathbf{H} = \mathbf{J}.$$

Here $\rho = \rho(x, y, z)$ and $\mathbf{J}(x, y, z)$ are assumed to be known. The radiation that the fields produce through a surface S is determined by a radiation flux density vector field, called the ***Poynting*** vector field,

$$\mathbf{P} = \mathbf{E} \times \mathbf{H}.$$

(a) If S is a *closed* surface, show that the radiation flux—that is, the flux of **P** through S—is given by

$$\iint_S \mathbf{P}\cdot d\mathbf{S} = -\iiint_V \mathbf{E}\cdot\mathbf{J}\,dV,$$

where V is the region enclosed by S.

(b) Examples of such fields are

$$\mathbf{E}(x, y, z) = z\mathbf{j} + y\mathbf{k} \qquad \text{and} \qquad \mathbf{H}(x, y, z) = -xy\mathbf{i} + x\mathbf{j} + yz\mathbf{k}.$$

In this case, find the flux of the Poynting vector through the hemispherical shell shown in Figure 8.5.3. (Notice that it is an *open* surface.)

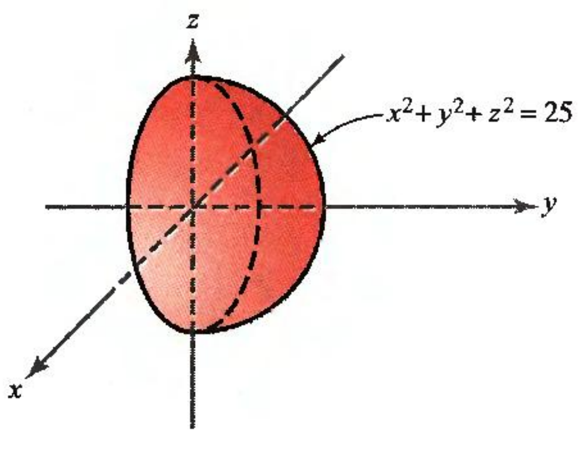

Figure 8.5.3 The surface for Exercise 10.

(c) The fields of part (b) produce a Poynting vector field passing through the toroidal surface shown in Figure 8.5.4. What is the flux through this torus?

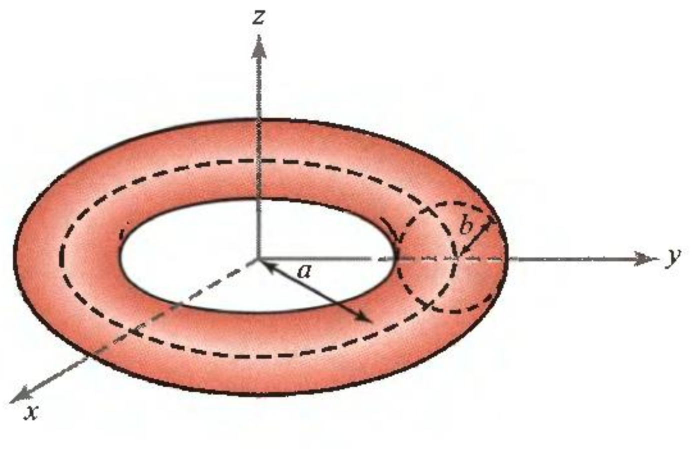

Figure 8.5.4 The surface for Exercise 10(c).

8.6 Differential Forms

The theory of differential forms provides an elegant way of formulating Green's, Stokes', and Gauss' theorems as one statement, the ***fundamental theorem of calculus***. The birth of the concept of a differential form is another dramatic example of how mathematics speaks to mathematicians and drives its own development. These three theorems are, in reality, generalizations of the fundamental theorem of calculus of Newton and Leibniz for functions of one variable,

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

to two and three dimensions.

Recall that Bernhard Riemann created the concept of n-dimensional spaces. If the fundamental theorem of calculus was truly *fundamental,* then it should generalize to arbitrary dimensions. But wait! The cross product, and therefore the curl, does not generalize to higher dimensions, as we remarked in footnote 3, in Section 1.3. Thus, some new idea is needed.

Recall that Hamilton searched for almost 15 years for his quaternions, which ultimately led to the discovery of the cross product. What is the nonexistence of a cross product in higher dimensions telling us? If the fundamental theorem of calculus is the core concept, this suggests the existence of a mathematical language in which it can be formulated in n-dimensions. In order to achieve this, mathematicians realized that they were forced to move away from vectors and on to the discovery of *dual* vectors and an entirely new mathematical object, a *differential form.* In this new language, all of the theorems of Green, Stokes, and Gauss have the same elegant and extraordinarily simple form.

Simply and very briefly stated, an expression of the type $P\,dx + Q\,dy$ is a 1-form, or a *differential one-form* on a region in the xy plane, and $F\,dx\,dy$ is a 2-form. Analogously, one can define the notion of an n-form. There is an operation d, which