8 Vector Analysis in Higher Dimensions

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Introduction

In this concluding chapter, our goal is to find a way to unify and extend the three main theorems of vector analysis (namely, the theorems of Green, Gauss, and Stokes). To accomplish such a task, we need to develop the notion of a **differential form** whose integral embraces and generalizes line, surface, and volume integrals.

8.1 An Introduction to Differential Forms

Throughout this section, U will denote an open set in $\mathbf{R}^n$, where $\mathbf{R}^n$ has coordinates $(x_1, x_2, \ldots, x_n)$, as usual. Any functions that appear are assumed to be appropriately differentiable.

Differential Forms

We begin by giving a new name to an old friend. If $f\colon U \subseteq \mathbf{R}^n \to \mathbf{R}$ is a scalar-valued function (of class C^k), we will also refer to f as a **differential 0-form**, or just a **0-form** for short. 0-forms can be added to one another and multiplied together, as well we know.

The next step is to describe differential 1-forms. Ultimately, we will see that a differential 1-form is a generalization of $f(x)\,dx$—that is, of something that can be integrated with respect to a single variable, such as with a line integral. More precisely, in $\mathbf{R}^n$, the **basic differential 1-forms** are denoted $dx_1, dx_2, \ldots, dx_n$. A general (**differential**) **1-form** ω is an expression that is built from the basic 1-forms as

$$\omega = F_1(x_1, \ldots, x_n)\,dx_1 + F_2(x_1, \ldots, x_n)\,dx_2 + \ldots + F_n(x_1, \ldots, x_n)\,dx_n,$$

where, for $j = 1, \ldots, n$, F_j is a scalar-valued function (of class C^k) on $U \subseteq \mathbf{R}^n$. Differential 1-forms can be added to one another, and we can multiply a 0-form f and a 1-form ω (both defined on $U \subseteq \mathbf{R}^n$) in the obvious way: If

$$\omega = F_1\,dx_1 + F_2\,dx_2 + \cdots + F_n\,dx_n,$$

then

$$f\omega = f F_1\,dx_1 + f F_2\,dx_2 + \cdots + f F_n\,dx_n.$$

EXAMPLE 1 In $\mathbf{R}^3$, let

$$\omega = xyz\,dx + z^2 \cos y\,dy + ze^x\,dz \quad \text{and} \quad \eta = (y - z)\,dx + z^2 \sin y\,dy - 2\,dz.$$

Then

$$\omega + \eta = (xyz + y - z)\,dx + z^2(\cos y + \sin y)\,dy + (ze^x - 2)\,dz.$$

If $f(x, y, z) = xe^y - z$, then

$$f\omega = (xe^y - z)xyz\,dx + (xe^y - z)z^2 \cos y\,dy + (xe^y - z)ze^x dz.$$ ◆

Thus far, we have described 1-forms merely as formal expressions in certain symbols. But 1-forms can also be thought of as functions. The basic 1-forms $dx_1, \ldots, dx_n$ take as argument a vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ in $\mathbf{R}^n$; the value of dx_i on $\mathbf{a}$ is

$$dx_i(\mathbf{a}) = a_i.$$

In others words, dx_i extracts the ith component of the vector $\mathbf{a}$.

More generally, for each $\mathbf{x}_0 \in U$, the 1-form ω gives rise to a combination $\omega_{\mathbf{x}_0}$ of basic 1-forms

$$\omega_{\mathbf{x}_0} = F_1(\mathbf{x}_0)\,dx_1 + \cdots + F_n(\mathbf{x}_0)\,dx_n;$$

$\omega_{\mathbf{x}_0}$ acts on the vector $\mathbf{a} \in \mathbf{R}^n$ as

$$\omega_{\mathbf{x}_0}(\mathbf{a}) = F_1(\mathbf{x}_0)\,dx_1(\mathbf{a}) + F_2(\mathbf{x}_0)\,dx_2(\mathbf{a}) + \cdots + F_n(\mathbf{x}_0)\,dx_n(\mathbf{a}).$$

EXAMPLE 2 Suppose ω is the 1-form defined on $\mathbf{R}^3$ by

$$\omega = x^2yz\,dx + y^2z\,dy - 3xyz\,dz.$$

If $\mathbf{x}_0 = (1, -2, 5)$ and $\mathbf{a} = (a_1, a_2, a_3)$, then

$$\begin{aligned} \omega_{(1,-2,5)}(\mathbf{a}) &= -10\,dx(\mathbf{a}) + 20\,dy(\mathbf{a}) + 30\,dz(\mathbf{a}) \\ &= -10a_1 + 20a_2 + 30a_3, \end{aligned}$$

and, if $\mathbf{x}_0 = (3, 4, 6)$, then

$$\begin{aligned} \omega_{(3,4,6)}(\mathbf{a}) &= 216\,dx(\mathbf{a}) + 96\,dy(\mathbf{a}) - 216\,dz(\mathbf{a}) \\ &= 216a_1 + 96a_2 - 216a_3. \end{aligned}$$

The notation suggests that a 1-form is a function of the vector $\mathbf{a}$ but that this function varies from point to point as $\mathbf{x}_0$ changes. Indeed, 1-forms are actually functions on vector fields. ◆

A **basic (differential) 2-form** on $\mathbf{R}^n$ is an expression of the form

$$dx_i \wedge dx_j, \quad i, j = 1, \ldots, n.$$

It is also a function that requires *two* vector arguments $\mathbf{a}$ and $\mathbf{b}$, and we evaluate this function as

$$dx_i \wedge dx_j(\mathbf{a}, \mathbf{b}) = \begin{vmatrix} dx_i(\mathbf{a}) & dx_i(\mathbf{b}) \\ dx_j(\mathbf{a}) & dx_j(\mathbf{b}) \end{vmatrix}.$$

(The determinant represents, up to sign, the area of the parallelogram spanned by the projections of $\mathbf{a}$ and $\mathbf{b}$ in the x_ix_j-plane.) It is not difficult to see that, for $i, j = 1, \ldots, n$,

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \tag{1}$$

and

$$dx_i \wedge dx_i = 0. \tag{2}$$

Formula (1) can be established by comparing $dx_i \wedge dx_j(\mathbf{a}, \mathbf{b})$ with $dx_j \wedge dx_i(\mathbf{a}, \mathbf{b})$. Formula (2) follows from formula (1). Given formulas (1) and (2), we see that we can generate all the linearly independent, nontrivial basic 2-forms on $\mathbf{R}^n$ by listing all possible terms $dx_i \wedge dx_j$, where i and j are integers between 1 and n with $i < j$:

$$\begin{gathered} dx_1 \wedge dx_2,\ dx_1 \wedge dx_3, \ldots, dx_1 \wedge dx_n, \\ dx_2 \wedge dx_3, \ldots, dx_2 \wedge dx_n, \\ \vdots \\ dx_{n-1} \wedge dx_n. \end{gathered}$$

To count how many 2-forms are in this list, note that there are n choices for dx_i and $n-1$ choices for dx_j (so that $dx_i \neq dx_j$ in view of (2)), and a "correction" factor of 2 so as not to count both $dx_i \wedge dx_j$ and $dx_j \wedge dx_i$ in light of (1). Hence, there are $n(n-1)/2$ independent 2-forms.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. A general (**differential**) **2-form** on $U \subseteq \mathbf{R}^n$ is an expression

$$\omega = F_{12}(\mathbf{x})\, dx_1 \wedge dx_2 + F_{13}(\mathbf{x})\, dx_1 \wedge dx_3 + \cdots + F_{n-1n}(\mathbf{x})\, dx_{n-1} \wedge dx_n,$$

where each F_{ij} is a real-valued function $F_{ij}: U \subseteq \mathbf{R}^n \to \mathbf{R}$. The idea here is to generalize something that can be integrated with respect to two variables—such as with a surface integral.

EXAMPLE 3 In $\mathbf{R}^3$, a general 2-form may be written as

$$F_1(x, y, z)\, dy \wedge dz + F_2(x, y, z)\, dz \wedge dx + F_3(x, y, z)\, dx \wedge dy.$$

The reason for using this somewhat curious ordering of the terms in the sum will, we hope, become clear later in the chapter. ◆

Given a point $\mathbf{x}_0 \in U \subseteq \mathbf{R}^n$, to evaluate a general 2-form on the ordered pair $(\mathbf{a}, \mathbf{b})$ of vectors, we have

$$\begin{aligned} \omega_{\mathbf{x}_0}(\mathbf{a}, \mathbf{b}) = {} & F_{12}(\mathbf{x}_0)\, dx_1 \wedge dx_2(\mathbf{a}, \mathbf{b}) + F_{13}(\mathbf{x}_0)\, dx_1 \wedge dx_3(\mathbf{a}, \mathbf{b}) \\ & + \cdots + F_{n-1n}(\mathbf{x}_0)\, dx_{n-1} \wedge dx_n(\mathbf{a}, \mathbf{b}). \end{aligned}$$

EXAMPLE 4 In $\mathbf{R}^3$, let $\omega = 3xy\, dy \wedge dz + (2y + z)\, dz \wedge dx + (x - z)\, dx \wedge dy$. Then

$$\begin{aligned} \omega_{(1,2,-3)}(\mathbf{a}, \mathbf{b}) &= 6\, dy \wedge dz(\mathbf{a}, \mathbf{b}) + dz \wedge dx(\mathbf{a}, \mathbf{b}) + 4\, dx \wedge dy(\mathbf{a}, \mathbf{b}) \\ &= 6 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + \begin{vmatrix} a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} + 4 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \\ &= 6(a_2 b_3 - a_3 b_2) + (a_3 b_1 - a_1 b_3) + 4(a_1 b_2 - a_2 b_1). \end{aligned}$$

◆

Finally, we generalize the notions of 1-forms and 2-forms to provide a definition of a ***k*-form**.

DEFINITION 1.1 Let k be a positive integer. A **basic (differential)** ***k*-form** on $\mathbf{R}^n$ is an expression of the form

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $1 \le i_j \le n$ for $j = 1, \ldots, k$. The basic k-forms are also functions that require k vector arguments $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ and are evaluated as

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = \det \begin{bmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \cdots & dx_{i_1}(\mathbf{a}_k) \\ dx_{i_2}(\mathbf{a}_1) & dx_{i_2}(\mathbf{a}_2) & \cdots & dx_{i_2}(\mathbf{a}_k) \\ \vdots & \vdots & \ddots & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \cdots & dx_{i_k}(\mathbf{a}_k) \end{bmatrix}.$$

EXAMPLE 5 Let

$$\mathbf{a}_1 = (1, 2, -1, 3, 0), \qquad \mathbf{a}_2 = (5, 4, 3, 2, 1), \qquad \text{and} \qquad \mathbf{a}_3 = (0, 1, 3, -2, 0)$$

be three vectors in $\mathbf{R}^5$. Then we have

$$dx_1 \wedge dx_3 \wedge dx_5(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \det \begin{bmatrix} 1 & 5 & 0 \\ -1 & 3 & 3 \\ 0 & 1 & 0 \end{bmatrix} = -3.$$ ◆

Using properties of determinants, we can show that

$$dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_k} = -dx_{i_1} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k} \tag{3}$$

and

$$dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \ldots \wedge dx_{i_k} = 0. \tag{4}$$

Formula (3) says that switching two terms (namely, dx_{i_j} and dx_{i_l}) in the basic k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ causes a sign change, and formula (4) says that a basic k-form containing two identical terms is zero. Formulas (3) and (4) generalize formulas (1) and (2).

DEFINITION 1.2 A general **(differential)** ***k*-form** on $U \subseteq \mathbf{R}^n$ is an expression of the form

$$\omega = \sum_{i_1, \ldots, i_k = 1}^{n} F_{i_1 \ldots i_k}(\mathbf{x})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where each $F_{i_1 \ldots i_k}$ is a real-valued function $F_{i_1 \ldots i_k} \colon U \to \mathbf{R}$. Given a point $\mathbf{x}_0 \in U$, we evaluate ω on an ordered k-tuple $(\mathbf{a}_1, \ldots, \mathbf{a}_k)$ of vectors as

$$\omega_{\mathbf{x}_0}(\mathbf{a}_1, \ldots, \mathbf{a}_k) = \sum_{i_1, \ldots, i_k = 1}^{n} F_{i_1 \ldots i_k}(\mathbf{x}_0)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \ldots, \mathbf{a}_k).$$

Note that a 0-form is so named because, in order to be consistent with a 1-form or 2-form, it must take zero vector arguments!

In view of formulas (3) and (4), we write a general k-form as

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} F_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

(That is, the sum may be taken over strictly increasing indices $i_1, \ldots, i_k$.) For example, the 4-form

$$\begin{aligned} \omega = {} & x_2\, dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 + (x_3 - x_5^2)\, dx_1 \wedge dx_2 \wedge dx_5 \wedge dx_3 \\ & + x_1 x_3\, dx_5 \wedge dx_3 \wedge dx_4 \wedge dx_1 \end{aligned}$$

may be written in the "standard form" with increasing indices as

$$\omega = (x_2 - x_1 x_3)\, dx_1 \wedge dx_3 \wedge dx_4 \wedge dx_5 + (x_5^2 - x_3)\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5.$$

Two k-forms may be added in the obvious way, and the product of a 0-form f and a k-form ω is analogous to the product of a 0-form and a 1-form.

Exterior Product

The symbol $\wedge$ that we have been using does, in fact, denote a type of multiplication called the **exterior** (or **wedge**) **product**. The exterior product can be extended to general differential forms in the following manner:

DEFINITION 1.3 Let $U \subseteq \mathbf{R}^n$ be open. Let f denote a 0-form on U. Let $\omega = \sum F_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ denote a k-form on U and $\eta = \sum G_{j_1 \ldots j_l} dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ an l-form. Then we define

$$f \wedge \omega = f\omega = \sum f F_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

$$\omega \wedge \eta = \sum F_{i_1 \ldots i_k} G_{j_1 \ldots j_l} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

Thus, the wedge product of a k-form and an l-form is a $(k + l)$-form.

EXAMPLE 6 Let

$$\omega = x_1^2\, dx_1 \wedge dx_2 + (2x_3 - x_2)\, dx_1 \wedge dx_3 + e^{x_3}\, dx_3 \wedge dx_4$$

and

$$\eta = x_4\, dx_1 \wedge dx_3 \wedge dx_5 + x_6\, dx_2 \wedge dx_4 \wedge dx_6$$

be, respectively, a 2-form and a 3-form on $\mathbf{R}^6$. Then Definition 1.3 yields

$$\begin{aligned} \omega \wedge \eta = {} & x_1^2 x_4\, dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_5 \\ & + (2x_3 - x_2) x_4\, dx_1 \wedge dx_3 \wedge dx_1 \wedge dx_3 \wedge dx_5 \\ & + e^{x_3} x_4\, dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_3 \wedge dx_5 \\ & + x_1^2 x_6\, dx_1 \wedge dx_2 \wedge dx_2 \wedge dx_4 \wedge dx_6 \\ & + (2x_3 - x_2) x_6\, dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4 \wedge dx_6 \\ & + e^{x_3} x_6\, dx_3 \wedge dx_4 \wedge dx_2 \wedge dx_4 \wedge dx_6. \end{aligned}$$

Because of formula (4), most of the terms in this sum are zero. In fact,

$$\begin{aligned}\omega \wedge \eta &= (2x_3 - x_2)x_6\, dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4 \wedge dx_6 \\ &= (x_2 - 2x_3)x_6\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_6,\end{aligned}$$

using formula (3). ◆

From the various definitions and observations made so far, we can establish the following results, which are useful when computing with differential forms:

PROPOSITION 1.4 (Properties of the exterior product) Assume that all the differential forms that follow are defined on $U \subseteq \mathbf{R}^n$:

1. **Distributivity.** If ω_1 and ω_2 are k-forms and η is an l-form, then

$$(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta.$$

2. **Anticommutativity.** If ω is a k-form and η an l-form, then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

3. **Associativity.** If ω is a k-form, η an l-form, and τ a p-form, then

$$(\omega \wedge \eta) \wedge \tau = \omega \wedge (\eta \wedge \tau).$$

4. **Homogeneity.** If ω is a k-form, η an l-form, and f a 0-form, then

$$(f\omega) \wedge \eta = f(\omega \wedge \eta) = \omega \wedge (f\eta).$$

8.1 Exercises

Determine the values of the following differential forms on the ordered sets of vectors indicated in Exercises 1–7.

1. $dx_1 - 3\, dx_2$; $\mathbf{a} = (7, 3)$

2. $2\, dx + 6\, dy - 5\, dz$; $\mathbf{a} = (1, -1, -2)$

3. $3\, dx_1 \wedge dx_2$; $\mathbf{a} = (4, -1)$, $\mathbf{b} = (2, 0)$

4. $4\, dx \wedge dy - 7\, dy \wedge dz$; $\mathbf{a} = (0, 1, -1)$, $\mathbf{b} = (1, 3, 2)$

5. $7\, dx \wedge dy \wedge dz$; $\mathbf{a} = (1, 0, 3)$, $\mathbf{b} = (2, -1, 0)$, $\mathbf{c} = (5, 2, 1)$

6. $dx_1 \wedge dx_2 + 2\, dx_2 \wedge dx_3 + 3\, dx_3 \wedge dx_4$; $\mathbf{a} = (1, 2, 3, 4)$, $\mathbf{b} = (4, 3, 2, 1)$

7. $2\, dx_1 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5$; $\mathbf{a} = (1, 0, -1, 4, 2)$, $\mathbf{b} = (0, 0, 9, 1, -1)$, $\mathbf{c} = (5, 0, 0, 0, -2)$

8. Let ω be the 1-form on $\mathbf{R}^3$ defined by

$$\omega = x^2 y\, dx + y^2 z\, dy + z^3 x\, dz.$$

Find $\omega_{(3,-1,4)}(\mathbf{a})$, where $\mathbf{a} = (a_1, a_2, a_3)$.

9. Let ω be the 2-form on $\mathbf{R}^4$ given by

$$\omega = x_1 x_3\, dx_1 \wedge dx_3 - x_2 x_4\, dx_2 \wedge dx_4.$$

Find $\omega_{(2,-1,-3,1)}(\mathbf{a}, \mathbf{b})$.

10. Let ω be the 2-form on $\mathbf{R}^3$ given by

$$\omega = \cos x\, dx \wedge dy - \sin z\, dy \wedge dz + (y^2 + 3) dx \wedge dz.$$

Find $\omega_{(0,-1,\pi/2)}(\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$.

11. Let ω be as in Exercise 10. Find $\omega_{(x,y,z)}((2, 0, -1), (1, 7, 5))$.

12. Let ω be the 3-form on $\mathbf{R}^3$ given by

$$\omega = \left(e^x \cos y + (y^2 + 2)e^{2z}\right) dx \wedge dy \wedge dz.$$

Find $\omega_{(0,0,0)}(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, and $\mathbf{c} = (c_1, c_2, c_3)$.

13. Let ω be as in Exercise 12. Find $\omega_{(x,y,z)}((1, 0, 0), (0, 2, 0), (0, 0, 3))$.

In Exercises 14–19, determine $\omega \wedge \eta$.

14. On $\mathbf{R}^3$: $\omega = 3\, dx + 2\, dy - x\, dz$; $\eta = x^2 dx - \cos y\, dy + 7\, dz$.

15. On $\mathbf{R}^3$: $\omega = y\, dx - x\, dy$; $\eta = z\, dx \wedge dy + y\, dx \wedge dz + x\, dy \wedge dz$.

16. On $\mathbf{R}^4$: $\omega = 2\,dx_1 \wedge dx_2 - x_3\,dx_2 \wedge dx_4$; $\eta = 2x_4\,dx_1 \wedge dx_3 + (x_3 - x_2)\,dx_3 \wedge dx_4$.

17. On $\mathbf{R}^4$: $\omega = x_1\,dx_1 + 2x_2\,dx_2 + 3x_3\,dx_3$; $\eta = (x_1 + x_2)\,dx_1 \wedge dx_2 \wedge dx_3 + (x_3 - x_4)\,dx_1 \wedge dx_2 \wedge dx_4$.

18. On $\mathbf{R}^4$: $\omega = (x_1 + x_2)\,dx_1 \wedge dx_2 \wedge dx_3 + (x_3 - x_4)\,dx_1 \wedge dx_2 \wedge dx_4$; $\eta = x_1\,dx_1 + 2x_2\,dx_2 + 3x_3\,dx_3$.

19. On $\mathbf{R}^5$: $\omega = x_1\,dx_2 \wedge dx_3 - x_2x_3\,dx_1 \wedge dx_5$; $\eta = e^{x_4x_5}dx_1 \wedge dx_4 \wedge dx_5 - x_1 \cos x_5\,dx_2 \wedge dx_3 \wedge dx_4$.

20. Prove formula (3) by evaluating $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$ on k vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ in $\mathbf{R}^n$.

21. Prove formula (4). (Hint: Use formula (3).)

22. Explain why a k-form on $\mathbf{R}^n$ with $k > n$ must be identically zero.

23. Prove property 1 of Proposition 1.4.

24. Prove property 2 of Proposition 1.4. (Hint: Use formula (3).)

25. Prove property 3 of Proposition 1.4.

26. Prove property 4 of Proposition 1.4.

8.2 Manifolds and Integrals of *k*-forms

In this section, we investigate how to integrate k-forms over k-dimensional objects (i.e., curves, surfaces, and higher-dimensional analogues) in $\mathbf{R}^n$.

Integrals over Curves and Surfaces

We begin by considering integrals of 1-forms and 2-forms over parametrized curves and surfaces.

DEFINITION 2.1 Let $\mathbf{x}\colon [a, b] \to \mathbf{R}^n$ be a C^1 path in $\mathbf{R}^n$. If ω is a 1-form defined on an open set $U \subseteq \mathbf{R}^n$ that contains the image of $\mathbf{x}$, then the **integral of ω over $\mathbf{x}$**, denoted $\int_{\mathbf{x}} \omega$, is

$$\int_{\mathbf{x}} \omega = \int_a^b \omega_{\mathbf{x}(t)}(\mathbf{x}'(t))\,dt.$$

EXAMPLE 1 Let $\omega = (x^2 + y)\,dx + yz\,dy + (x + y - z)\,dz$. We integrate ω over the path $\mathbf{x}\colon [0, 1] \to \mathbf{R}^3$, $\mathbf{x}(t) = (2t + 3, 3t, 7 - t)$.

We have $\mathbf{x}'(t) = (2, 3, -1)$ so that, using Definition 2.1, we find that

$$\begin{aligned}\int_{\mathbf{x}} \omega &= \int_0^1 \omega_{(2t+3,3t,7-t)}(2, 3, -1)\,dt \\ &= \int_0^1 [((2t + 3)^2 + 3t)\,dx(2, 3, -1) + 3t(7 - t)\,dy(2, 3, -1) \\ &\qquad + (2t + 3 + 3t - (7 - t))\,dz(2, 3, -1)]\,dt \\ &= \int_0^1 [(4t^2 + 15t + 9)\cdot 2 + (21t - 3t^2)\cdot 3 + (6t - 4)\cdot(-1)]\,dt \\ &= \int_0^1 (-t^2 + 87t + 22)\,dt = \tfrac{391}{6}.\end{aligned}$$

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In general, if

$$\omega = F_1\,dx_1 + F_2\,dx_2 + \cdots + F_n\,dx_n$$

is a 1-form on $\mathbf{R}^n$ and $\mathbf{x}\colon [a, b] \to \mathbf{R}^n$ is any path, then

$$\begin{aligned}\omega_{\mathbf{x}(t)}(\mathbf{x}'(t)) &= F_1(\mathbf{x}(t))\, dx_1(\mathbf{x}'(t)) + F_2(\mathbf{x}(t))\, dx_2(\mathbf{x}'(t)) \\ &\quad + \cdots + F_n(\mathbf{x}(t))\, dx_n(\mathbf{x}'(t)) \\ &= F_1(\mathbf{x}(t))x_1'(t) + F_2(\mathbf{x}(t))x_2'(t) + \cdots + F_n(\mathbf{x}(t))x_n'(t) \\ &= (F_1(\mathbf{x}(t)), F_2(\mathbf{x}(t)), \ldots, F_n(\mathbf{x}(t))) \cdot \mathbf{x}'(t).\end{aligned}$$

From this we conclude the following:

PROPOSITION 2.2 If $\mathbf{F}$ denotes the vector field $(F_1, F_2, \ldots, F_n)$ and

$$\omega = F_1\, dx_1 + F_2\, dx_2 + \cdots + F_n\, dx_n$$

and if $\mathbf{x}\colon [a, b] \to \mathbf{R}^n$ is a C^1 path, then

$$\int_{\mathbf{x}} \omega = \int_a^b \omega_{\mathbf{x}(t)}(\mathbf{x}'(t))\, dt = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}.$$

That is, integrating a 1-form over a path (or, indeed, over a simple, piecewise C^1 curve) is exactly the same as computing a vector line integral.

Now we see how to integrate 2-forms over parametrized surfaces in $\mathbf{R}^3$.

DEFINITION 2.3 Let D be a bounded, connected region in $\mathbf{R}^2$ and let $\mathbf{X}\colon D \to \mathbf{R}^3$ be a smooth parametrized surface in $\mathbf{R}^3$. If ω is a 2-form defined on an open set in $\mathbf{R}^3$ that contains $\mathbf{X}(D)$, then we define $\int_{\mathbf{X}} \omega$, **the integral of ω over $\mathbf{X}$**, as

$$\int_{\mathbf{X}} \omega = \iint_D \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t)\, ds\, dt.$$

(Recall that $\mathbf{T}_s = \partial\mathbf{X}/\partial s$ and $\mathbf{T}_t = \partial\mathbf{X}/\partial t$.)

Let's work out the integral in Definition 2.3. We write ω as

$$F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$$

and $\mathbf{X}(s, t)$ as $(x(s, t), y(s, t), z(s, t))$. Therefore,

$$\begin{aligned}\int_{\mathbf{X}} \omega &= \iint_D \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t)\, ds\, dt \\ &= \iint_D [F_1(\mathbf{X}(s, t))\, dy \wedge dz(\mathbf{T}_s, \mathbf{T}_t) + F_2(\mathbf{X}(s, t))\, dz \wedge dx(\mathbf{T}_s, \mathbf{T}_t) \\ &\qquad + F_3(\mathbf{X}(s, t))\, dx \wedge dy(\mathbf{T}_s, \mathbf{T}_t)]\, ds\, dt.\end{aligned}$$

By definition of the basic 2-forms,

$$\begin{aligned}dy \wedge dz(\mathbf{T}_s, \mathbf{T}_t) &= \det \begin{bmatrix} dy(\mathbf{T}_s) & dy(\mathbf{T}_t) \\ dz(\mathbf{T}_s) & dz(\mathbf{T}_t) \end{bmatrix} \\ &= \det \begin{bmatrix} \partial y/\partial s & \partial y/\partial t \\ \partial z/\partial s & \partial z/\partial t \end{bmatrix} = \frac{\partial(y, z)}{\partial(s, t)}.\end{aligned}$$

Similarly, we have

$$dz \wedge dx(\mathbf{T}_s, \mathbf{T}_t) = \frac{\partial(z, x)}{\partial(s, t)} \quad \text{and} \quad dx \wedge dy(\mathbf{T}_s, \mathbf{T}_t) = \frac{\partial(x, y)}{\partial(s, t)}.$$

Hence, if $\mathbf{F} = (F_1, F_2, F_3)$, then

$$\begin{aligned}\int_{\mathbf{X}} \omega &= \iint_D \left[F_1(\mathbf{X}(s,t)) \frac{\partial(y, z)}{\partial(s, t)} + F_2(\mathbf{X}(s,t)) \frac{\partial(z, x)}{\partial(s, t)} \right.\\ &\qquad \left. + F_3(\mathbf{X}(s,t)) \frac{\partial(x, y)}{\partial(s, t)} \right] ds\, dt \\ &= \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \left(\frac{\partial(y, z)}{\partial(s, t)}, \frac{\partial(z, x)}{\partial(s, t)}, \frac{\partial(x, y)}{\partial(s, t)} \right) ds\, dt.\end{aligned}$$

Recall from formula (7) in §7.1 that

$$\left(\frac{\partial(y, z)}{\partial(s, t)}, \frac{\partial(z, x)}{\partial(s, t)}, \frac{\partial(x, y)}{\partial(s, t)} \right) = \mathbf{N}(s, t),$$

the normal to $\mathbf{X}(D)$ at the point $\mathbf{X}(s, t)$. Therefore, we have established the following Proposition (see also Definition 2.2 of Chapter 7):

PROPOSITION 2.4 If $\mathbf{F}$ denotes the vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$ and

$$\omega = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$$

and if $\mathbf{X}\colon D \to \mathbf{R}^3$ is a smooth parametrized surface such that ω (or $\mathbf{F}$) is defined on an open set containing $\mathbf{X}(D)$, then

$$\begin{aligned}\int_{\mathbf{X}} \omega = \iint_D \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t)\, ds\, dt &= \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt \\ &= \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}.\end{aligned}$$

Parametrized Manifolds

Next, we generalize the notions of parametrized curves and surfaces to higher-dimensional objects in $\mathbf{R}^n$. To set notation, let $\mathbf{R}^k$ have coordinates $(u_1, u_2, \ldots, u_k)$.

DEFINITION 2.5 Let D be a region in $\mathbf{R}^k$ that consists of an open, connected set, possibly together with some or all of its boundary points. A **parametrized k-manifold** in $\mathbf{R}^n$ is a continuous map $\mathbf{X}\colon D \to \mathbf{R}^n$ that is one-one except, possibly, along ∂D. We refer to the image $M = \mathbf{X}(D)$ as the **underlying manifold of $\mathbf{X}$** (or the manifold **parametrized by $\mathbf{X}$**).

Such a k-manifold possesses k **coordinate curves** defined from $\mathbf{X}$ by holding all the variables $u_1, \ldots, u_k$ fixed except one; namely, the jth coordinate curve is the curve parametrized by

$$u_j \longmapsto \mathbf{X}(a_1, \ldots, a_{j-1}, u_j, a_{j+1}, \ldots, a_k),$$

where the a_i's $(i \neq j)$ are fixed constants. If $\mathbf{X}$ is differentiable and $x_1, x_2, \ldots, x_n$ denote the component functions of $\mathbf{X}$, then the tangent vector to the jth coordinate curve, denoted $\mathbf{T}_{u_j}$, is

$$\mathbf{T}_{u_j} = \frac{\partial \mathbf{X}}{\partial u_j} = \left(\frac{\partial x_1}{\partial u_j}, \frac{\partial x_2}{\partial u_j}, \ldots, \frac{\partial x_n}{\partial u_j} \right).$$

A parametrized k-manifold is said to be **smooth** at a point $\mathbf{X}(\mathbf{u}_0)$ if the mapping $\mathbf{X}$ is of class C^1 in a neighborhood of $\mathbf{u}_0$ and if the k tangent vectors $\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_k}$ are linearly independent at $\mathbf{X}(\mathbf{u}_0)$. (Recall that k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbf{R}^n$ are **linearly independent** if the equation $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ holds if and only if $c_1 = c_2 = \cdots = c_k = 0$.) A parametrized k-manifold is said to be **smooth** if it is smooth at every point of $\mathbf{X}(\mathbf{u}_0)$ with $\mathbf{u}_0$ in the interior of D.

Sometimes we will refer to the underlying manifold $M = \mathbf{X}(D)$ of a parametrized manifold $\mathbf{X}: D \to \mathbf{R}^n$ as a parametrized manifold; we do not expect any confusion will result from this abuse of terminology.

EXAMPLE 2 Let $D = [0, 1] \times [1, 2] \times [-1, 1]$ and $\mathbf{X}: D \to \mathbf{R}^5$ be given by

$$\mathbf{X}(u_1, u_2, u_3) = (u_1 + u_2, 3u_2, u_2u_3^2, u_2 - u_3, 5u_3).$$

We show that $M = \mathbf{X}(D)$ is a smooth parametrized 3-manifold in $\mathbf{R}^5$.

Note first that $\mathbf{X}$ is continuous (in fact, of class C^∞) since its component functions are polynomials. To see that $\mathbf{X}$ is one-one, consider the equation

$$\mathbf{X}(\mathbf{u}) = \mathbf{X}(\tilde{\mathbf{u}}); \tag{1}$$

we show that $\mathbf{u} = \tilde{\mathbf{u}}$. Equation (1) is equivalent to a system of five equations:

$$\begin{cases} u_1 + u_2 = \tilde{u}_1 + \tilde{u}_2 \\ 3u_2 = 3\tilde{u}_2 \\ u_2u_3^2 = \tilde{u}_2\tilde{u}_3^2 \\ u_2 - u_3 = \tilde{u}_2 - \tilde{u}_3 \\ 5u_3 = 5\tilde{u}_3 \end{cases}.$$

The second equation implies $u_2 = \tilde{u}_2$, and the last equation implies $u_3 = \tilde{u}_3$. Hence, the first equation becomes

$$u_1 + u_2 = \tilde{u}_1 + u_2 \quad \Longleftrightarrow \quad u_1 = \tilde{u}_1.$$

Thus,

$$\mathbf{u} = (u_1, u_2, u_3) = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \tilde{\mathbf{u}}.$$

To check the smoothness of M, note that the tangent vectors to the three coordinate curves are

$$\mathbf{T}_{u_1} = \frac{\partial \mathbf{X}}{\partial u_1} = (1, 0, 0, 0, 0);$$

$$\mathbf{T}_{u_2} = \frac{\partial \mathbf{X}}{\partial u_2} = (1, 3, u_3^2, 1, 0);$$

$$\mathbf{T}_{u_3} = \frac{\partial \mathbf{X}}{\partial u_3} = (0, 0, 2u_2u_3, -1, 5).$$

Therefore, to have $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 = \mathbf{0}$, we must have

$$(c_1 + c_2, 3c_2, u_3^2 c_2 + 2u_2u_3c_3, c_2 - c_3, 5c_3) = (0, 0, 0, 0, 0).$$

It readily follows that $c_1 = c_2 = c_3 = 0$ is the only possibility for a solution. Hence, $\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}$ are linearly independent at all $\mathbf{u} \in D$ and so M is smooth at all points. ◆

Parametrized k-manifolds, although seemingly abstract mathematical notions when k is larger than 3, are actually very useful for describing a variety of situations, one of which is illustrated in the next example.

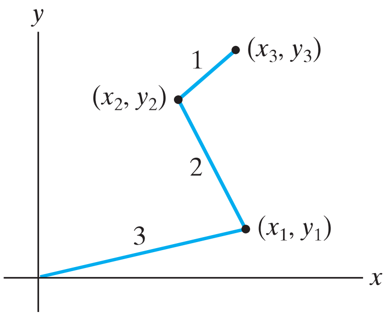

Figure 8.1 The planar robot arm of Example 3. Each rod is free to pivot about the appropriate linkage points.

EXAMPLE 3 A planar robot arm is constructed consisting of three linked rods of lengths 1, 2, and 3. (See Figure 8.1.) The rod of length 3 is anchored at the origin of $\mathbf{R}^2$ but free to rotate about the origin. The rod of length 2 is attached to the free end of the rod of length 3, and the rod of length 1 is, in turn, attached to the free end of the rod of length 2. We describe the set of positions that the arm can take as a parametrized manifold.

Clearly, each state of the robot arm is determined by the coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3) of the linkage points, which we may consider to form a vector $\mathbf{x} = (x_1, y_1, x_2, y_2, x_3, y_3)$ in $\mathbf{R}^6$. However, not all vectors in $\mathbf{R}^6$ represent a state of the robot arm. In particular, the point (x_1, y_1) must lie on the circle of radius 3, centered at the origin, the point (x_2, y_2) must lie on the circle of radius 2, centered at (x_1, y_1), and the point (x_3, y_3) must lie on the circle of radius 1, centered at (x_2, y_2). Thus, for $\mathbf{x} = (x_1, y_1, x_2, y_2, x_3, y_3)$ to represent a state of the robot arm, we require

$$\begin{cases} x_1^2 + y_1^2 = 9 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = 4 \,. \\ (x_3 - x_2)^2 + (y_3 - y_2)^2 = 1 \end{cases} \qquad (2)$$

We may parametrize each of the circles in the system (2) in a one-one fashion by using three different angles θ_1, θ_2, and θ_3. Hence, we find

$$\begin{aligned} (x_1, y_1) &= (3\cos\theta_1, 3\sin\theta_1), \\ (x_2, y_2) &= (x_1 + 2\cos\theta_2, y_1 + 2\sin\theta_2) \\ &= (3\cos\theta_1 + 2\cos\theta_2, 3\sin\theta_1 + 2\sin\theta_2), \end{aligned} \qquad (3)$$

and

$$\begin{aligned} (x_3, y_3) &= (x_2 + \cos\theta_3, y_2 + \sin\theta_3) \\ &= (3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3, 3\sin\theta_1 + 2\sin\theta_2 + \sin\theta_3), \end{aligned}$$

where $0 \le \theta_1, \theta_2, \theta_3 < 2\pi$. Therefore, the map $\mathbf{X}\colon [0, 2\pi) \times [0, 2\pi) \times [0, 2\pi) \to \mathbf{R}^6$ given by

$$\mathbf{X}(\theta_1, \theta_2, \theta_3) = (x_1, y_1, x_2, y_2, x_3, y_3),$$

where $(x_1, y_1, x_2, y_2, x_3, y_3)$ are given in terms of θ_1, θ_2, and θ_3 by means of the equations in (3), exhibits the set of states of the robot arm as a parametrized 3-manifold in $\mathbf{R}^6$. We leave it to you to check that $\mathbf{X}$ defines a smooth parametrized 3-manifold. ◆

Just like a parametrized surface, a parametrized k-manifold $M = \mathbf{X}(D)$ may or may not have a **boundary**, denoted, as usual, by ∂M. If M has a nonempty boundary, then ∂M is contained in the image under $\mathbf{X}$ of the portion of the boundary of the domain region D that is also part of D. Under suitable (and mild) hypotheses, ∂M, if nonempty, is, in turn, a union of finitely many $(k-1)$-manifolds (*without* boundaries).

EXAMPLE 4 Let $B \subset \mathbf{R}^3$ denote the closed unit ball $\{\mathbf{u} = (u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 \leq 1\}$, and define $\mathbf{X}: B \to \mathbf{R}^4$ by

$$\mathbf{X}(u_1, u_2, u_3) = (u_1, u_2, u_3, u_1^2 + u_2^2 + u_3^2).$$

Then $M = \mathbf{X}(B)$ is a portion of a "generalized paraboloid" having equation $w = x^2 + y^2 + z^2$; we have $M = \{(x, y, z, w) \in \mathbf{R}^4 \mid w = x^2 + y^2 + z^2,\ x^2 + y^2 + z^2 \leq 1\}$. In this case, $\partial M = \{(x, y, z, 1) \mid x^2 + y^2 + z^2 = 1\}$. Note that ∂M is a parametrized 2-manifold in $\mathbf{R}^4$, as we may see via the map

$$\mathbf{Y}: [0, \pi] \times [0, 2\pi) \to \mathbf{R}^4, \qquad \mathbf{Y}(s, t) = (\sin s \cos t, \sin s \sin t, \cos s, 1). \quad ◆$$

Integrals over Parametrized k-manifolds

Now, we see how to define the integral of a k-form over a smooth parametrized k-manifold. Our definition generalizes those of Definitions 2.1 and 2.3.

DEFINITION 2.6 Let D be a bounded, connected region in $\mathbf{R}^k$ and $\mathbf{X}: D \to \mathbf{R}^n$ a smooth parametrized k-manifold. If ω is a k-form defined on an open set in $\mathbf{R}^n$ that contains $M = \mathbf{X}(D)$, then we define the **integral** of ω over M (denoted $\int_{\mathbf{X}} \omega$) by

$$\int_{\mathbf{X}} \omega = \int \cdots \int_D \omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_k})\, du_1 \cdots du_k.$$

(Here $\int \cdots \int$ refers to the k-dimensional integral over D.)

EXAMPLE 5 Let $\mathbf{X}: [0, 1] \times [1, 2] \times [-1, 1] \to \mathbf{R}^5$ be the parametrized 3-manifold defined by

$$\mathbf{X}(u_1, u_2, u_3) = (u_1 + u_2, 3u_2, u_2 u_3^2, u_2 - u_3, 5u_3).$$

(See Example 2.) Let ω be the 3-form defined on $\mathbf{R}^5$ as

$$\omega = x_1 x_3\, dx_1 \wedge dx_3 \wedge dx_5 + (x_3 x_4 - 2x_2 x_5)\, dx_2 \wedge dx_4 \wedge dx_5.$$

We calculate $\int_{\mathbf{X}} \omega$.

Recall from Example 2 that the tangent vectors to the three coordinate curves are

$$\mathbf{T}_{u_1} = (1, 0, 0, 0, 0),$$

$$\mathbf{T}_{u_2} = (1, 3, u_3^2, 1, 0),$$

and

$$\mathbf{T}_{u_3} = (0, 0, 2u_2 u_3, -1, 5).$$

Then, from Definition 2.6,

$$\int_{\mathbf{X}} \omega$$

$$= \int_{-1}^{1}\int_{1}^{2}\int_{0}^{1} \{(u_1+u_2)u_2u_3^2\, dx_1 \wedge dx_3 \wedge dx_5(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3})$$

$$+ (u_2u_3^2(u_2-u_3) - 30u_2u_3)\, dx_2 \wedge dx_4 \wedge dx_5(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3})\}\, du_1\, du_2\, du_3$$

$$= \int_{-1}^{1}\int_{1}^{2}\int_{0}^{1} \left\{ (u_1+u_2)u_2u_3^2 \begin{vmatrix} 1 & 1 & 0 \\ 0 & u_3^2 & 2u_2u_3 \\ 0 & 0 & 5 \end{vmatrix} \right.$$

$$\left. + (u_2^2u_3^2 - u_2u_3^3 - 30u_2u_3) \begin{vmatrix} 0 & 3 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{vmatrix} \right\} du_1\, du_2\, du_3$$

$$= \int_{-1}^{1}\int_{1}^{2}\int_{0}^{1} 5(u_1+u_2)u_2u_3^4\, du_1\, du_2\, du_3 = \tfrac{37}{6}.$$ ◆

EXAMPLE 6 If ω is a 3-form on $\mathbf{R}^3$, then ω may be written as

$$\omega = F(x, y, z)\, dx \wedge dy \wedge dz.$$

(Why?) If D^* is a bounded region in $\mathbf{R}^3$ and $\mathbf{X}\colon D^* \to \mathbf{R}^3$ is a smooth parametrized 3-manifold, then Definition 2.6 tells us that

$$\int_{\mathbf{X}} \omega = \iiint_{D^*} \omega_{\mathbf{X}(u_1,u_2,u_3)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3})\, du_1\, du_2\, du_3$$

$$= \iiint_{D^*} F(\mathbf{X}(\mathbf{u}))\, dx \wedge dy \wedge dz(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3})\, du_1\, du_2\, du_3$$

$$= \iiint_{D^*} F(\mathbf{X}(\mathbf{u})) \begin{vmatrix} \partial x/\partial u_1 & \partial x/\partial u_2 & \partial x/\partial u_3 \\ \partial y/\partial u_1 & \partial y/\partial u_2 & \partial y/\partial u_3 \\ \partial z/\partial u_1 & \partial z/\partial u_2 & \partial z/\partial u_3 \end{vmatrix} du_1\, du_2\, du_3$$

$$= \iiint_{D^*} F(x(\mathbf{u}), y(\mathbf{u}), z(\mathbf{u})) \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)}\, du_1\, du_2\, du_3$$

$$= \pm \iiint_{D} F(x, y, z)\, dx\, dy\, dz,$$

from the change of variables theorem for triple integrals (Theorem 5.5 of Chapter 5), where $D = \mathbf{X}(D^*)$. ◆

Orientation of a Parametrized *k*-manifold

We have seen that vector line integrals and vector surface integrals may be defined, respectively, over oriented curves and surfaces in a manner effectively independent of the parametrization used. We now see how it is possible to define the integral

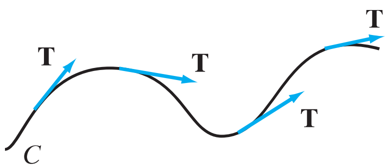

Figure 8.2 An orientation of the curve C shown is a choice of continuously varying unit tangent vector $\mathbf{T}$ along C.

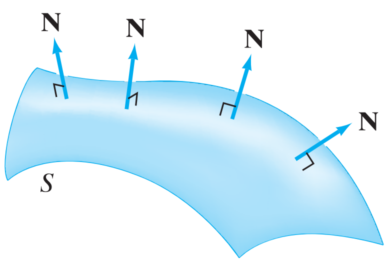

Figure 8.3 An orientation of the surface S is a choice of continuously varying unit normal vector $\mathbf{N}$ along S.

of a k-form over a parametrized k-manifold $\mathbf{X}: D \to \mathbf{R}^n$ so that it depends largely on the underlying manifold $M = \mathbf{X}(D)$, rather than on the particular map $\mathbf{X}$. To do this, we must consider how **reparametrization** of M affects the integral, and we must define what we mean by an **orientation** of M.

First, we consider the notion of orientation. We have previously seen how parametrized curves and surfaces can be oriented by using some fairly natural geometric ideas. A smooth parametrized curve implicitly received an orientation from the parameter; typically, we orient a curve by indicating the direction in which the parameter variable increases. We may also think of an **orientation** of a curve as a choice of a unit tangent vector $\mathbf{T}$ at each point of the curve, made so that $\mathbf{T}$ varies continuously as we move along the curve. (See Figure 8.2.) An **orientation** of a smooth parametrized surface in $\mathbf{R}^3$, when it exists, is a choice of a continuously varying unit normal vector $\mathbf{N}$ at each point of the surface. (See Figure 8.3.)

To define notions of **orientation** and **orientability** for a parametrized k-manifold when $k > 2$, we will need to work more formally.

First, we need to introduce two related ideas from the linear algebra of $\mathbf{R}^n$. Thus, suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in $\mathbf{R}^n$. By a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, we mean any vector $\mathbf{v} \in \mathbf{R}^n$ that can be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

for suitable choices of the scalars $c_1, \ldots, c_k$. The set of all possible linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, called the (**linear**) **span** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, will be denoted $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. That is,

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k \mid c_1, \ldots, c_k \in \mathbf{R}\}.$$

DEFINITION 2.7 Let $M = \mathbf{X}(D)$, where $\mathbf{X}: D \subseteq \mathbf{R}^k \to \mathbf{R}^n$, be a smooth parametrized k-manifold. An **orientation** of M is a choice of a smooth, nonzero k-form Ω defined on M. If such a k-form Ω exists, M is said to be **orientable** and **oriented** once a choice of such a k-form is made.

Although we cannot readily visualize an orientation Ω of a parametrized k-manifold when k is large, we can nonetheless see how the tangent vectors to the coordinate curves relate to it.

DEFINITION 2.8 Let $M = \mathbf{X}(D)$ be a smooth parametrized k-manifold oriented by the k-form Ω. The tangent vectors $\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_k}$ to the coordinate curves of M are said to be **compatible** with Ω if

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_k}) > 0.$$

We also say that the parametrization $\mathbf{X}$ is **compatible** with the orientation Ω if the corresponding tangent vectors $\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_k}$ are.

Note that if $\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_k}$ are incompatible with the orientation Ω, then they are compatible with the *opposite* orientation $-\Omega$. Alternatively, we may change the parametrization $\mathbf{X}$ of M by reordering the variables $u_1, \ldots, u_k$ to, say, $u_2, u_1, u_3, \ldots, u_k$, so that $\mathbf{T}_{u_2}, \mathbf{T}_{u_1}, \mathbf{T}_{u_3}, \ldots, \mathbf{T}_{u_k}$ are compatible with Ω.

Definition 2.7 is consistent with the earlier definitions of orientations of curves and surfaces, as we now discuss. Suppose first that $\mathbf{x}\colon I \to \mathbf{R}^n$ is a smooth parametrized curve in $\mathbf{R}^n$ (where I is an interval in $\mathbf{R}$) and $\mathbf{T}$ is a continuously varying choice of unit tangent vector along $C = \mathbf{x}(I)$. Then we may define an orientation 1-form Ω on C by

$$\Omega_{\mathbf{x}(t)}(\mathbf{a}) = \mathbf{T} \cdot \mathbf{a}.$$

Conversely, given an orientation 1-form Ω, we may define a continuously varying unit tangent vector from it by taking $\mathbf{T}$ to be the unique unit vector parallel to $\mathbf{x}'(t)$ such that, for any nonzero vector $\mathbf{a}$ parallel to $\mathbf{x}'(t)$,

$$\mathbf{T} \cdot \mathbf{a} \text{ has the same sign as } \Omega_{\mathbf{x}(t)}(\mathbf{a}).$$

That $\mathbf{T}$ is uniquely determined follows because $\mathbf{T}$ must equal $\pm\mathbf{x}'(t)/\|\mathbf{x}'(t)\|$, so knowing $\mathbf{a}$ and the value of $\Omega_{\mathbf{x}(t)}(\mathbf{a})$ determines the choice of sign for $\mathbf{T}$.

Similarly, suppose $S = \mathbf{X}(D)$ is a smooth parametrized surface in $\mathbf{R}^3$ (i.e., a smooth parametrized 2-manifold). If we can orient S by a continuously varying unit normal $\mathbf{N}$, then we may define an orientation 2-form Ω on S by

$$\Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{a}, \mathbf{b}) = \det\begin{bmatrix} \mathbf{N} & \mathbf{a} & \mathbf{b} \end{bmatrix},$$

where $\begin{bmatrix} \mathbf{N} & \mathbf{a} & \mathbf{b} \end{bmatrix}$ is the 3×3 matrix whose columns are, in order, the vectors $\mathbf{N}$, $\mathbf{a}$, $\mathbf{b}$. Conversely, given an orientation 2-form Ω on S, we may define a continuously varying unit normal $\mathbf{N}$ from it by taking $\mathbf{N}$ to be the *unique* unit vector perpendicular to $\mathbf{T}_{u_1}$ and $\mathbf{T}_{u_2}$ (and hence to every vector in $\operatorname{Span}\{\mathbf{T}_{u_1}, \mathbf{T}_{u_2}\}$) such that, for any pair $\mathbf{a}$, $\mathbf{b}$ of linearly independent vectors in $\operatorname{Span}\{\mathbf{T}_{u_1}, \mathbf{T}_{u_2}\}$,

$$\det\begin{bmatrix} \mathbf{N} & \mathbf{a} & \mathbf{b} \end{bmatrix} \text{ has the same sign as } \Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{a}, \mathbf{b}).$$

To see that $\mathbf{N}$ is uniquely determined, note that, given linearly independent vectors $\mathbf{a}$, $\mathbf{b}$ in $\operatorname{Span}\{\mathbf{T}_{u_1}, \mathbf{T}_{u_2}\}$, the only possibilities for $\mathbf{N}$ are

$$\pm\frac{\mathbf{T}_{u_1} \times \mathbf{T}_{u_2}}{\|\mathbf{T}_{u_1} \times \mathbf{T}_{u_2}\|}.$$

Hence, we choose the sign for the normal vector $\mathbf{N}$ so that $\det\begin{bmatrix} \mathbf{N} & \mathbf{a} & \mathbf{b} \end{bmatrix}$ has the same sign as $\Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{a}, \mathbf{b})$.

EXAMPLE 7 Consider the generalized paraboloid $M = \{(x, y, z, w) \in \mathbf{R}^4 \mid w = x^2 + y^2 + z^2\}$, which we may exhibit as a smooth parametrized 3-manifold via

$$\mathbf{X}\colon \mathbf{R}^3 \to \mathbf{R}^4, \qquad \mathbf{X}(u_1, u_2, u_3) = (u_1, u_2, u_3, u_1^2 + u_2^2 + u_3^2).$$

We show how to orient M.

Note that the equation $x^2 + y^2 + z^2 - w = 0$ shows that M is the level set at height 0 of the function $F(x, y, z, w) = x^2 + y^2 + z^2 - w$. Hence, the gradient $\nabla F = (2x, 2y, 2z, -1)$ is a vector normal to M. If we employ the parametrization $\mathbf{X}$ and normalize the (parametrized) gradient, we see that

$$\mathbf{N}(u_1, u_2, u_3) = \frac{(2u_1, 2u_2, 2u_3, -1)}{\sqrt{4u_1^2 + 4u_2^2 + 4u_3^2 + 1}}$$

is a continuously varying unit normal. Moreover, the 3-form Ω defined on M as

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \det\begin{bmatrix} \mathbf{N} & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

gives an orientation for M. Note that

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}) = \det\begin{bmatrix}\mathbf{N} & \mathbf{T}_{u_1} & \mathbf{T}_{u_2} & \mathbf{T}_{u_3}\end{bmatrix}$$

$$= \det\begin{bmatrix} \dfrac{2u_1}{\sqrt{4u_1^2+4u_2^2+4u_3^2+1}} & 1 & 0 & 0 \\ \dfrac{2u_2}{\sqrt{4u_1^2+4u_2^2+4u_3^2+1}} & 0 & 1 & 0 \\ \dfrac{2u_3}{\sqrt{4u_1^2+4u_2^2+4u_3^2+1}} & 0 & 0 & 1 \\ \dfrac{-1}{\sqrt{4u_1^2+4u_2^2+4u_3^2+1}} & 2u_1 & 2u_2 & 2u_3 \end{bmatrix}$$

$$= \sqrt{4u_1^2+4u_2^2+4u_3^2+1}.$$

Since this last expression is strictly positive, we see that $\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}$ are compatible with Ω. ◆

EXAMPLE 8 We may generalize Example 7 as follows:

Suppose that $M \subseteq \mathbf{R}^n$ is the graph of a function $f: U \subseteq \mathbf{R}^{n-1} \to \mathbf{R}^n$; that is, suppose M is defined by the equation $x_n = f(x_1, \ldots, x_{n-1})$. Then M may be parametrized as an $(n-1)$-manifold via

$$\mathbf{X}: U \subseteq \mathbf{R}^{n-1} \to \mathbf{R}^n, \quad \mathbf{X}(u_1, \ldots, u_{n-1}) = (u_1, \ldots, u_{n-1}, f(u_1, \ldots, u_{n-1})).$$

Since M is also the level set at height 0 of the function

$$F(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1}) - x_n,$$

a vector normal to M is provided by the gradient $\nabla F = (f_{x_1}, \ldots, f_{x_{n-1}}, -1)$. If we normalize ∇F and use the parametrization $\mathbf{X}$, we see that we have a continuously varying unit normal

$$\mathbf{N}(u_1, \ldots, u_{n-1}) = \frac{(f_{u_1}, \ldots, f_{u_{n-1}}, -1)}{\sqrt{f_{u_1}^2 + \cdots + f_{u_{n-1}}^2 + 1}},$$

from which we may define our orientation $(n-1)$-form Ω for M by

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}) = \det\begin{bmatrix}\mathbf{N} & \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1}\end{bmatrix}.$$ ◆

Now suppose that M is a smooth parametrized k-manifold in $\mathbf{R}^n$ with nonempty boundary ∂M. If M is oriented by the k-form Ω, then there is a way to derive from it an orientation for ∂M, which we describe in Definition 2.9. To set notation, let $\mathbf{X}: D \subseteq \mathbf{R}^k \to \mathbf{R}^n$ denote the parametrization of M and suppose $\mathbf{Y}: E \subseteq \mathbf{R}^{k-1} \to \mathbf{R}^n$ gives a parametrization of a connected piece of ∂M as a smooth $(k-1)$-manifold. Since ∂M is part of M, if $\mathbf{s} = (s_1, \ldots, s_{k-1}) \in E$, then there is some $\mathbf{u} = (u_1, \ldots, u_k) \in D$ such that $\mathbf{Y}(\mathbf{s}) = \mathbf{X}(\mathbf{u})$.

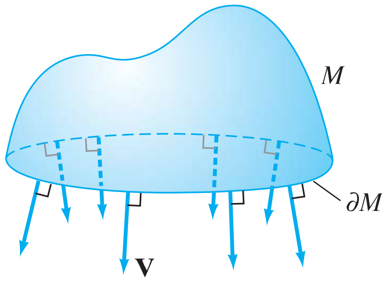

Figure 8.4 The outward-pointing unit vector **V** of Definition 2.9.

DEFINITION 2.9 Let M be a smooth parametrized k-manifold in $\mathbf{R}^n$ with boundary ∂M. Suppose M is oriented by the k-form Ω. Then the connected pieces of ∂M are said to be **oriented consistently** with M, or that ∂M has its **orientation induced** from that of M, if the orientation $(k-1)$-form $\Omega^{\partial M}$ is determined from Ω as follows. Let **V** be the unique, outward-pointing unit vector in $\mathbf{R}^n$, defined and varying continuously along ∂M, that is tangent to M and normal to ∂M. (See Figure 8.4.) Then $\Omega^{\partial M}$ is defined as

$$\Omega^{\partial M}_{\mathbf{Y}(\mathbf{s})}(\mathbf{a}_1, \ldots, \mathbf{a}_{k-1}) = \Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{V}, \mathbf{a}_1, \ldots, \mathbf{a}_{k-1}),$$

where the map $\mathbf{X}\colon D \subseteq \mathbf{R}^k \to \mathbf{R}^n$ parametrizes M, the map $\mathbf{Y}\colon E \subseteq \mathbf{R}^{k-1} \to \mathbf{R}^n$ parametrizes a connected piece of ∂M, and $\mathbf{Y}(\mathbf{s}) = \mathbf{X}(\mathbf{u})$.

Note that, in particular, the vector **V** in Definition 2.9 must be such that

- $\mathbf{V} \in \operatorname{Span}\{\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_k}\}$ (i.e., **V** is tangent to M);
- $\mathbf{V} \cdot \mathbf{T}_{s_i} = 0$ for $i = 1, \ldots, k-1$ (i.e., **V** is normal to ∂M);
- **V** points away from M.

These conditions are often not difficult to achieve in practice. Definition 2.9 will be very important when we consider a generalization of Stokes's theorem in the next section.

EXAMPLE 9 Consider the surface S in $\mathbf{R}^3$ consisting of the portion of the cylinder $x^2 + y^2 = 4$ with $2 \le z \le 5$. Note that the boundary of S consists of the two circles $\{(x, y, z) \mid x^2 + y^2 = 4,\ z = 2\}$ and $\{(x, y, z) \mid x^2 + y^2 = 4,\ z = 5\}$. We investigate how to orient ∂S consistently with an orientation of S.

The cylinder may be parametrized as a 2-manifold in $\mathbf{R}^3$ by

$$\mathbf{X}\colon [0, 2\pi) \times [2, 5] \to \mathbf{R}^3, \qquad \mathbf{X}(u_1, u_2) = (2\cos u_1, 2\sin u_1, u_2).$$

Then the tangent vectors to the coordinate curves are

$$\mathbf{T}_{u_1} = (-2\sin u_1, 2\cos u_1, 0)$$

and

$$\mathbf{T}_{u_2} = (0, 0, 1).$$

Since S is a portion of the level set at height 4 of the function $F(x, y, z) = x^2 + y^2$, a unit normal **N** to S is given by

$$\frac{\nabla F}{\|\nabla F\|} = \frac{(2x, 2y, 0)}{\sqrt{4x^2 + 4y^2}} = \left(\frac{x}{2}, \frac{y}{2}, 0\right).$$

In terms of the parametrization **X**, the normal **N** is also given by

$$\mathbf{N} = (\cos u_1, \sin u_1, 0).$$

Then we may define an orientation 2-form on S by

$$\Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{a}_1, \mathbf{a}_2) = \det\begin{bmatrix}\mathbf{N} & \mathbf{a}_1 & \mathbf{a}_2\end{bmatrix}.$$

Hence,

$$\Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det \begin{bmatrix} \cos u_1 & -2\sin u_1 & 0 \\ \sin u_1 & 2\cos u_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2 > 0.$$

Thus, $\mathbf{T}_{u_1}, \mathbf{T}_{u_2}$ are compatible with Ω.

We may parametrize ∂S by using two mappings:

$$\text{Bottom circle:} \quad \mathbf{Y}_1\colon [0, 2\pi) \to \mathbf{R}^3, \quad \mathbf{Y}_1(s) = (2\cos s, 2\sin s, 2)$$

and

$$\text{Top circle:} \quad \mathbf{Y}_2\colon [0, 2\pi) \to \mathbf{R}^3, \quad \mathbf{Y}_2(s) = (2\cos s, 2\sin s, 5)$$

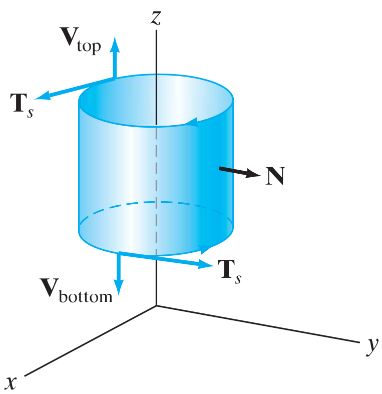

Figure 8.5 Orienting the boundary of the surface S of Example 9. Note the outward-pointing tangent vectors $\mathbf{V}_{\text{top}}$ and $\mathbf{V}_{\text{bottom}}$.

To use Definition 2.9 to orient ∂S, we must identify outward-pointing vectors tangent to S and normal to ∂S. From Figure 8.5, we see that along the top circle $\mathbf{V} = \mathbf{V}_{\text{top}} = (0, 0, 1)$ works, while along the bottom circle, $\mathbf{V} = \mathbf{V}_{\text{bottom}} = (0, 0, -1)$ suffices. Hence, Definition 2.9 tells us that, along the bottom circle,

$$\Omega^{\partial S}_{\mathbf{Y}_1(s)}(\mathbf{a}) = \Omega_{\mathbf{X}(s,2)}(\mathbf{V}_{\text{bottom}}, \mathbf{a}) = \det \begin{bmatrix} \mathbf{N} & \mathbf{V}_{\text{bottom}} & \mathbf{a} \end{bmatrix},$$

while along the top circle,

$$\Omega^{\partial S}_{\mathbf{Y}_1(s)}(\mathbf{a}) = \Omega_{\mathbf{X}(s,5)}(\mathbf{V}_{\text{top}}, \mathbf{a}) = \det \begin{bmatrix} \mathbf{N} & \mathbf{V}_{\text{top}} & \mathbf{a} \end{bmatrix}.$$

For both maps $\mathbf{Y}_1$ and $\mathbf{Y}_2$, we have that the coordinate tangent vector is $\mathbf{T}_s = (-2\sin s, 2\cos s, 0)$. Thus, along the bottom circle,

$$\Omega^{\partial S}_{\mathbf{Y}_1(s)}(\mathbf{T}_s) = \det \begin{bmatrix} \cos s & 0 & -2\sin s \\ \sin s & 0 & 2\cos s \\ 0 & -1 & 0 \end{bmatrix} = 2,$$

so $\mathbf{T}_s$ is compatible with the orientation 1-form $\Omega^{\partial S}$. However, along the top circle,

$$\Omega^{\partial S}_{\mathbf{Y}_2(s)}(\mathbf{T}_s) = \det \begin{bmatrix} \cos s & 0 & -2\sin s \\ \sin s & 0 & 2\cos s \\ 0 & 1 & 0 \end{bmatrix} = -2,$$

so $\mathbf{T}_s$ is incompatible with $\Omega^{\partial S}$. Therefore, we must orient the top circle *clockwise* around the z-axis and the bottom circle *counterclockwise*. ◆

The following example is the three-dimensional analogue of Example 9:

EXAMPLE 10 Consider the subset $M \subseteq \mathbf{R}^4$ given by $M = \{(x, y, z, w) \mid x^2 + y^2 + z^2 = 4,\ 2 \le w \le 5\}$. This set M is a portion of the cylinder over a sphere of radius 2. Note that the boundary of M consists of the two spheres $S_{\text{bottom}} = \{(x, y, z, 2) \mid x^2 + y^2 + z^2 = 4\}$ and $S_{\text{top}} = \{(x, y, z, 5) \mid x^2 + y^2 + z^2 = 4\}$. We investigate M and ∂M as parametrized manifolds, orient M, and study the induced orientation on ∂M.

First, we note that M may be parametrized as a 3-manifold in $\mathbf{R}^4$ by

$$\mathbf{X}\colon [0, \pi] \times [0, 2\pi) \times [2, 5] \to \mathbf{R}^4,$$

$$\mathbf{X}(u_1, u_2, u_3) = (2\sin u_1 \cos u_2, 2\sin u_1 \sin u_2, 2\cos u_1, u_3).$$

(This is the usual parametrization of a sphere using spherical coordinates $\varphi = u_1$, $\theta = u_2$, with an additional parameter u_3 for the "vertical" w-axis.) The tangent vectors to the coordinate curves are given by

$$\mathbf{T}_{u_1} = (2\cos u_1 \cos u_2, 2\cos u_1 \sin u_2, -2\sin u_1, 0),$$
$$\mathbf{T}_{u_2} = (-2\sin u_1 \sin u_2, 2\sin u_1 \cos u_2, 0, 0),$$

and

$$\mathbf{T}_{u_3} = (0, 0, 0, 1).$$

Note that this parametrization fails to be smooth when u_1 is 0 or π, since then $\mathbf{T}_{u_2} = \mathbf{0}$ at those values for u_1. You can check that the parametrization is smooth at all other values of $\mathbf{X}(\mathbf{u})$ (i.e., for $\mathbf{u}$ in $(0, \pi) \times [0, 2\pi) \times [2, 5]$).

Because M is a portion of the level set at height 4 of the function $F(x, y, z, w) = x^2 + y^2 + z^2$, a unit normal $\mathbf{N}$ to M is given by

$$\frac{\nabla F}{\|\nabla F\|} = \frac{(2x, 2y, 2z, 0)}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, 0\right).$$

In terms of the parametrization $\mathbf{X}$, the normal $\mathbf{N}$ is also given by

$$\mathbf{N} = (\sin u_1 \cos u_2, \sin u_1 \sin u_2, \cos u_1, 0).$$

We define an orientation 3-form Ω for M by

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \det\left[\mathbf{N} \quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3\right].$$

Then

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}) = \det\begin{bmatrix} \sin u_1 \cos u_2 & 2\cos u_1 \cos u_2 & -2\sin u_1 \sin u_2 & 0 \\ \sin u_1 \sin u_2 & 2\cos u_1 \sin u_2 & 2\sin u_1 \cos u_2 & 0 \\ \cos u_1 & -2\sin u_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= 4\sin u_1 > 0$$

for $0 < u_1 < \pi$ (which is where the parametrization $\mathbf{X}$ is smooth). Hence, $\mathbf{T}_{u_1}$, $\mathbf{T}_{u_2}$, $\mathbf{T}_{u_3}$ are compatible with Ω.

We parametrize the two pieces of ∂M with two mappings:

"Bottom" sphere S_{bottom}:

$$\mathbf{Y}_1 \colon [0, \pi] \times [0, 2\pi) \to \mathbf{R}^4,$$
$$\mathbf{Y}_1(s_1, s_2) = (2\sin s_1 \cos s_2, 2\sin s_1 \sin s_2, 2\cos s_1, 2),$$

and

"Top" sphere S_{top}:

$$\mathbf{Y}_2 \colon [0, \pi] \times [0, 2\pi) \to \mathbf{R}^4,$$
$$\mathbf{Y}_2(s_1, s_2) = (2\sin s_1 \cos s_2, 2\sin s_1 \sin s_2, 2\cos s_1, 5).$$

Note that both parametrizations $\mathbf{Y}_1$ and $\mathbf{Y}_2$ give the same tangent vectors to the corresponding coordinate curves, namely,

$$\mathbf{T}_{s_1} = (2\cos s_1 \cos s_2, 2\cos s_1 \sin s_2, -2\sin s_1, 0)$$

and

$$\mathbf{T}_{s_2} = (-2\sin s_1 \sin s_2, 2\sin s_1 \cos s_2, 0, 0),$$

and, by considering these tangent vectors, we see that the parametrizations are smooth whenever $s_1 \neq 0, \pi$.

To give ∂M the orientation induced from that of M, we identify outward-pointing unit vectors tangent to M and normal to ∂M. Thus, we need $\mathbf{V}$ such that

- $\mathbf{V} \in \operatorname{Span}\{\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}\} \iff \mathbf{V} \cdot \mathbf{N} = 0$;
- $\mathbf{V} \cdot \mathbf{T}_{s_1} = \mathbf{V} \cdot \mathbf{T}_{s_2} = 0$;
- $\mathbf{V}$ points away from M.

It's not difficult to see that we must take $\mathbf{V} = \mathbf{V}_{\text{top}} = (0, 0, 0, 1)$ along S_{top} and $\mathbf{V} = \mathbf{V}_{\text{bottom}} = (0, 0, 0, -1)$ along S_{bottom}. Therefore, Definition 2.9 tells us that along S_{bottom},

$$\Omega^{\partial M}_{\mathbf{Y}_1(\mathbf{s})}(\mathbf{a}_1, \mathbf{a}_2) = \Omega_{\mathbf{X}(\mathbf{s},2)}(\mathbf{V}_{\text{bottom}}, \mathbf{a}_1, \mathbf{a}_2).$$

In particular,

$$\begin{aligned}
\Omega^{\partial M}_{\mathbf{Y}_1(\mathbf{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2}) &= \det\begin{bmatrix}\mathbf{N} & \mathbf{V}_{\text{bottom}} & \mathbf{T}_{s_1} & \mathbf{T}_{s_2}\end{bmatrix} \\
&= \det\begin{bmatrix} \sin s_1 \cos s_2 & 0 & 2\cos s_1 \cos s_2 & -2\sin s_1 \sin s_2 \\ \sin s_1 \sin s_2 & 0 & 2\cos s_1 \sin s_2 & 2\sin s_1 \cos s_2 \\ \cos s_1 & 0 & -2\sin s_1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \\
&= 4\sin s_1 > 0
\end{aligned}$$

for $0 < s_1 < \pi$ (i.e., where the parametrization $\mathbf{Y}_1$ is smooth). Thus, $\mathbf{Y}_1$ is compatible with $\Omega^{\partial M}$. Along S_{top}, however, we have

$$\begin{aligned}
\Omega^{\partial M}_{\mathbf{Y}_2(\mathbf{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2}) &= \det\begin{bmatrix}\mathbf{N} & \mathbf{V}_{\text{bottom}} & \mathbf{T}_{s_1} & \mathbf{T}_{s_2}\end{bmatrix} \\
&= \det\begin{bmatrix} \sin s_1 \cos s_2 & 0 & 2\cos s_1 \cos s_2 & -2\sin s_1 \sin s_2 \\ \sin s_1 \sin s_2 & 0 & 2\cos s_1 \sin s_2 & 2\sin s_1 \cos s_2 \\ \cos s_1 & 0 & -2\sin s_1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= -4\sin s_1 < 0
\end{aligned}$$

for $0 < s_1 < \pi$, so $\mathbf{Y}_2$ is incompatible with $\Omega^{\partial M}$. We must take care with this distinction when we consider the general version of Stokes's theorem. ◆

Next, we examine how the integral of a k-form ω can vary when taken over two different parametrizations $\mathbf{X}\colon D_1 \to \mathbf{R}^n$ and $\mathbf{Y}\colon D_2 \to \mathbf{R}^n$ for the same k-manifold $M = \mathbf{X}(D_1) = \mathbf{Y}(D_2)$.

DEFINITION 2.10 Let $\mathbf{X}\colon D_1 \subseteq \mathbf{R}^k \to \mathbf{R}^n$ and $\mathbf{Y}\colon D_2 \subseteq \mathbf{R}^k \to \mathbf{R}^n$ be parametrized k-manifolds. We say that $\mathbf{Y}$ is a **reparametrization** of $\mathbf{X}$ if there is a one-one and onto function $\mathbf{H}\colon D_2 \to D_1$ with inverse $\mathbf{H}^{-1} D_1 \to D_2$ such that $\mathbf{Y}(\mathbf{s}) = \mathbf{X}(\mathbf{H}(\mathbf{s}))$, that is, such that $\mathbf{Y} = \mathbf{X} \circ \mathbf{H}$. If $\mathbf{X}$ and $\mathbf{Y}$ are smooth and $\mathbf{H}$ and $\mathbf{H}^{-1}$ are both class C^1, then we say that $\mathbf{Y}$ is a **smooth** reparametrization of $\mathbf{X}$.

Since $\mathbf{H}$ is one-one, it can be shown that the Jacobian $\det D\mathbf{H}$ cannot change sign from positive to negative (or vice versa). Thus, we say that both $\mathbf{H}$ and $\mathbf{Y}$ are **orientation-preserving** if the Jacobian $\det D\mathbf{H}$ is positive, **orientation-reversing** if $\det D\mathbf{H}$ is negative.

The following result is a generalization of Theorem 2.5 of Chapter 7 to the case of k-manifolds.

THEOREM 2.11 Let $\mathbf{X}\colon D_1 \subseteq \mathbf{R}^k \to \mathbf{R}^n$ be a smooth parametrized k-manifold and ω a k-form defined on $\mathbf{X}(D_1)$. If $\mathbf{Y}\colon D_2 \subseteq \mathbf{R}^k \to \mathbf{R}^n$ is any smooth reparametrization of $\mathbf{X}$, then either

$$\int_{\mathbf{Y}} \omega = \int_{\mathbf{X}} \omega,$$

if $\mathbf{Y}$ is orientation-preserving, or

$$\int_{\mathbf{Y}} \omega = -\int_{\mathbf{X}} \omega,$$

if $\mathbf{Y}$ is orientation-reversing.

In view of Theorem 2.11, we can define what we mean by $\int_M \omega$, where M is a subset of $\mathbf{R}^n$ that can be parametrized as an oriented k-manifold and ω is a k-form defined on M. We simply let

$$\int_M \omega = \int_{\mathbf{X}} \omega,$$

where $\mathbf{X}\colon D \subseteq \mathbf{R}^k \to \mathbf{R}^n$ is *any* smooth parametrization of M that is compatible with the orientation chosen.

EXAMPLE 11 We evaluate $\int_C \omega$, where C is the (oriented) line segment in $\mathbf{R}^3$ from $(0, -1, -2)$ to $(1, 2, 3)$ and $\omega = z\,dx + x\,dy + y\,dz$.

Using Theorem 2.11, we may parametrize C in any way that preserves the orientation. Thus,

$$\mathbf{x}\colon [0, 1] \to \mathbf{R}^3, \quad \mathbf{x}(t) = (1-t)(0, -1, -2) + t(1, 2, 3) = (t, 3t-1, 5t-2)$$

is one way to make such a parametrization. Then $\mathbf{x}'(t) = (1, 3, 5)$ and, hence, from Definition 2.1, we have

$$\begin{aligned}
\int_C \omega = \int_{\mathbf{x}} \omega &= \int_0^1 \omega_{\mathbf{x}(t)}(\mathbf{x}'(t))\,dt \\
&= \int_0^1 \{(5t-2)\cdot 1 + t\cdot 3 + (3t-1)\cdot 5\}\,dt \\
&= \int_0^1 (23t-7)\,dt = \left(\frac{23}{2}t^2 - 7t\right)\bigg|_0^1 = \frac{9}{2}.
\end{aligned}$$

Note that if we parametrize C in the opposite direction by using, for example, the map

$$\mathbf{y}\colon [0, 1] \to \mathbf{R}^3, \quad \mathbf{y}(t) = t(0, -1, -2) + (1-t)(1, 2, 3) = (1-t, 2-3t, 3-5t),$$

then we would have

$$\int_{\mathbf{y}} \omega = \int_0^1 \omega_{\mathbf{y}(t)}(\mathbf{y}'(t))\,dt$$
$$= \int_0^1 \{(3-5t)(-1) + (1-t)(-3) + (2-3t)(-5)\}\,dt$$
$$= \int_0^1 (23t-16)\,dt = \left(\frac{23}{2}t^2 - 16t\right)\Big|_0^1 = -\frac{9}{2}.$$

In light of Theorem 2.11, this result could have been anticipated from our preceding calculation of $\int_{\mathbf{x}} \omega$. ◆

Note on *k*-manifolds

The central geometric object of study in this section, namely, a parametrized k-manifold, is actually a rather special case of a more general notion of a k-manifold. In general, a ***k*-manifold** in $\mathbf{R}^n$ is a connected subset $M \subseteq \mathbf{R}^n$ such that, for every point $\mathbf{x} \in M$, there is an open set $U \subseteq \mathbf{R}^k$ and a continuous, one-one map $\mathbf{X}: U \to \mathbf{R}^n$ with $\mathbf{x} \in \mathbf{X}(U) \subset M$. (A k-manifold with nonempty boundary requires a somewhat modified definition.) That is, M is a (general) k-manifold if it is *locally* a parametrized k-manifold near each point. It is possible to extend notions of orientation and integration of k-forms to this more general setting, although it requires some finesse to do so. For the types of examples we are encountering, however, our more restrictive definitions suffice.

8.2 Exercises

1. Check that the parametrized 3-manifold in Example 3 is in fact a smooth parametrized 3-manifold.

2. A planar robot arm is constructed by using two rods as shown in Figure 8.6. Suppose that each of the two rods may *telescope,* that is, that their respective lengths l_1 and l_2 may vary between 1 and 3 units. Show that the set of states of this robot arm may be described by a smooth parametrized 4-manifold in $\mathbf{R}^4$.

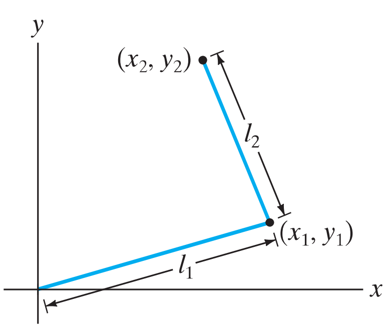

Figure 8.6 Figure for Exercise 2.

3. A planar robot arm is constructed by using a rod of length 3 anchored at the origin and two telescoping rods whose respective lengths l_2 and l_3 may vary between 1 and 2 units as shown in Figure 8.7. Show that the set of states of this robot arm may be described by a smooth parametrized 5-manifold in $\mathbf{R}^6$. (See Exercise 2.)

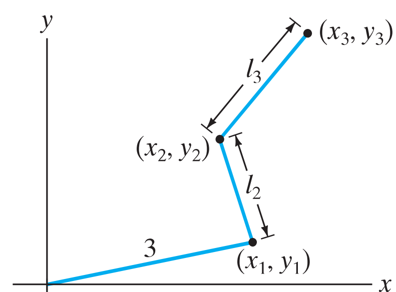

Figure 8.7 Figure for Exercise 3.

4. A robot arm is constructed in $\mathbf{R}^3$ by anchoring a rod of length 2 to the origin (using a ball joint so that the rod may swivel freely) and attaching to the free end of the rod another rod of length 1 (which may also swivel freely; see Figure 8.8). Show that the set of states of this robot arm may be described by a smooth parametrized 4-manifold in $\mathbf{R}^6$.

5. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are vectors in $\mathbf{R}^n$. If $\mathbf{x} \in \mathbf{R}^n$ is orthogonal to $\mathbf{v}_i$ for $i = 1, \ldots, k$, show that $\mathbf{x}$ is also orthogonal to any vector in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.

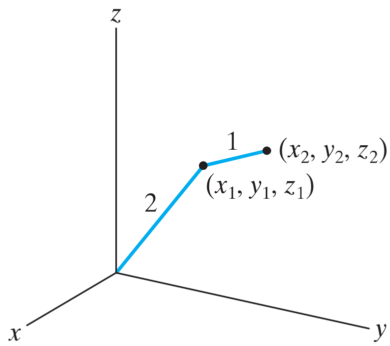

Figure 8.8 Figure for Exercise 4.

6. Let a, b, and c be positive constants and $\mathbf{x}\colon [0, \pi] \to \mathbf{R}^3$ the smooth path given by $\mathbf{x}(t) = (a\cos t, b\sin t, ct)$. If $\omega = b\,dx - a\,dy + xy\,dz$, calculate $\int_{\mathbf{x}} \omega$.

7. Evaluate $\int_C \omega$, where C is the unit circle $x^2 + y^2 = 1$, oriented counterclockwise, and $\omega = y\,dx - x\,dy$.

8. Compute $\int_C \omega$, where C is the line segment in $\mathbf{R}^n$ from $(0, 0, \ldots, 0)$ to $(3, 3, \ldots, 3)$ and $\omega = x_1\,dx_1 + x_2^2\,dx_2 + \cdots + x_n^n\,dx_n$.

9. Evaluate the integral $\int_{\mathbf{X}} \omega$, where $\mathbf{X}$ is the parametrized helicoid

$$\mathbf{X}(s, t) = (s\cos t, s\sin t, t), \qquad 0 \le s \le 1, 0 \le t \le 4\pi$$

and

$$\omega = z\,dx \wedge dy + 3\,dz \wedge dx - x\,dy \wedge dz.$$

10. Consider the helicoid parametrized as

$$\mathbf{X}(u_1, u_2) = (u_1\cos 3u_2, u_1\sin 3u_2, 5u_2),$$
$$0 \le u_1 \le 5, 0 \le u_2 \le 2\pi.$$

Let S denote the underlying surface of the helicoid and let Ω be the orientation 2-form defined in terms of $\mathbf{X}$ as

$$\Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{a}, \mathbf{b}) = \det \begin{bmatrix} -5\sin 3u_2 & a_1 & b_1 \\ 5\cos 3u_2 & a_2 & b_2 \\ -3u_1 & a_3 & b_3 \end{bmatrix}.$$

(a) Explain why the parametrization $\mathbf{X}$ is incompatible with Ω.

(b) Modify the parametrization $\mathbf{X}$ to one having the same underlying surface S but that is compatible with Ω.

(c) Alternatively, modify the orientation 2-form Ω to Ω' so that the original parametrization $\mathbf{X}$ is compatible with Ω'.

(d) Calculate $\int_S \omega$, where $\omega = z\,dx \wedge dy - (x^2 + y^2)\,dy \wedge dz$ and S is oriented using Ω.

11. Let M be the subset of $\mathbf{R}^3$ given by $\{(x, y, z) \mid x^2 + y^2 - 6 \le z \le 4 - x^2 - y^2\}$. Then M may be parametrized as a 3-manifold via

$$\mathbf{X}\colon D \to \mathbf{R}^3; \quad \mathbf{X}(u_1, u_2, u_3) = (u_1\cos u_2, u_1\sin u_2, u_3),$$

where

$$D = \{(u_1, u_2, u_3) \in \mathbf{R}^3 \mid 0 \le u_1 \le \sqrt{5},\ 0 \le u_2 < 2\pi,$$
$$u_1^2 - 6 \le u_3 \le 4 - u_1^2\}.$$

(The parameters u_1, u_2, and u_3 correspond, respectively, to the cylindrical coordinates r, θ, and z. Hence, it is straightforward to obtain the aforementioned parametrization.)

(a) Orient M by using the 3-form Ω, where

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \det\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix}.$$

Show that the parametrization, when smooth, is compatible with this orientation.

(b) Identify ∂M and parametrize it as a union of two 2-manifolds (i.e., as a piecewise smooth surface).

(c) Describe the outward-pointing unit vector $\mathbf{V}$, varying continuously along each smooth piece of ∂M, that is normal to ∂M. Give formulas for it in terms of the parametrizations used in part (b).

12. Calculate $\int_S \omega$, where S is the portion of the paraboloid $z = x^2 + y^2$ with $0 \le z \le 4$, oriented by upward-pointing normal vector $(-2x, -2y, 1)$, and $\omega = e^z dx \wedge dy + y\,dz \wedge dx + x\,dy \wedge dz$.

13. Calculate $\int_S \omega$, where S is the portion of the cylinder $x^2 + z^2 = 4$ with $-1 \le y \le 3$, oriented by outward normal vector $(x, 0, z)$, and $\omega = z\,dx \wedge dy + e^{y^2} dz \wedge dx + x\,dy \wedge dz$.

14. Consider the parametrized 2-manifold

$$\mathbf{X}\colon [1, 3] \times [0, 2\pi) \to \mathbf{R}^4, \quad \mathbf{X}(s, t) = (\sqrt{s}\cos t, \sqrt{4 - s}\sin t, \sqrt{s}\sin t, \sqrt{4 - s}\cos t).$$

Find

$$\int_{\mathbf{X}} (x_2^2 + x_4^2)\,dx_1 \wedge dx_3 - (2x_1^2 + 2x_3^2)\,dx_2 \wedge dx_4.$$

15. Consider the parametrized 3-manifold

$$\mathbf{X}\colon [0, 1] \times [0, 1] \times [0, 1] \to \mathbf{R}^4,$$
$$\mathbf{X}(u_1, u_2, u_3) = (u_1, u_2, u_3, (2u_1 - u_3)^2).$$

Find

$$\int_{\mathbf{X}} x_2\,dx_2 \wedge dx_3 \wedge dx_4 + 2x_1x_3\,dx_1 \wedge dx_2 \wedge dx_3.$$

8.3 The Generalized Stokes's Theorem

We conclude with a discussion of a generalization of Stokes's theorem that relates the integral of a k-form over a k-manifold to the integral of a $(k-1)$-form over the boundary of the manifold. Before we may state the result, however, we need to introduce the notion of the **exterior derivative** of a k-form.

The Exterior Derivative

The exterior derivative is an operator, denoted d, that takes differential k-forms to $(k+1)$-forms and is defined as follows:

DEFINITION 3.1 The **exterior derivative** df of a 0-form f on $U \subseteq \mathbf{R}^n$ is the 1-form

$$df = \frac{\partial f}{\partial x_1}\,dx_1 + \frac{\partial f}{\partial x_2}\,dx_2 + \cdots + \frac{\partial f}{\partial x_n}\,dx_n.$$

For $k > 0$, the **exterior derivative** of a k-form

$$\omega = \sum F_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

is the $(k+1)$-form

$$d\omega = \sum (dF_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $dF_{i_1 \ldots i_k}$ is computed as the exterior derivative of a 0-form.

EXAMPLE 1 If

$$f(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_2x_3 + x_4x_5x_6,$$

then

$$df = x_2x_3\,dx_1 + x_1x_3\,dx_2 + x_1x_2\,dx_3 + x_5x_6\,dx_4 + x_4x_6\,dx_5 + x_4x_5\,dx_6. \quad \blacklozenge$$

EXAMPLE 2 If ω is the 1-form

$$\omega = x_1x_2\,dx_1 + x_2x_3\,dx_2 + (2x_1 - x_2)\,dx_3,$$

then

$$\begin{aligned} d\omega &= d(x_1x_2) \wedge dx_1 + d(x_2x_3) \wedge dx_2 + d(2x_1 - x_2) \wedge dx_3 \\ &= (x_2\,dx_1 + x_1\,dx_2) \wedge dx_1 + (x_3\,dx_2 + x_2\,dx_3) \wedge dx_2 + (2dx_1 - dx_2) \wedge dx_3. \end{aligned}$$

Using the distributivity property in Proposition 1.4 and the facts that $dx_i \wedge dx_i = 0$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, we have

$$\begin{aligned} d\omega &= x_1\,dx_2 \wedge dx_1 + x_2\,dx_3 \wedge dx_2 + 2\,dx_1 \wedge dx_3 - dx_2 \wedge dx_3 \\ &= -x_1\,dx_1 \wedge dx_2 + 2\,dx_1 \wedge dx_3 - (x_2 + 1)\,dx_2 \wedge dx_3. \end{aligned} \quad \blacklozenge$$

Stokes's Theorem for *k*-forms

We now can state a generalization of Stokes's theorem to smooth parametrized k-manifolds in $\mathbf{R}^n$.

THEOREM 3.2 (GENERALIZED STOKES'S THEOREM) Let $D \subseteq \mathbf{R}^k$ be a closed, bounded, connected region, and let $M = \mathbf{X}(D)$ be an oriented, parametrized k-manifold in $\mathbf{R}^n$. If $\partial M \neq \emptyset$, let ∂M be given the orientation induced from that of M. Let ω denote a $(k-1)$-form defined on an open set in $\mathbf{R}^n$ that contains M. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

If $\partial M = \emptyset$, then we take $\int_{\partial M} \omega$ to be 0 in the preceding equation.

We make no attempt to prove Theorem 3.2.[1] Instead, we content ourselves for the moment by checking its correctness in a particular instance.

EXAMPLE 3 We verify the generalized Stokes's theorem (Theorem 3.2) for the 2-form $\omega = zw\, dx \wedge dy$, where M is the 3-manifold $M = \{(x, y, z, w) \in \mathbf{R}^4 \mid w = x^2 + y^2 + z^2,\ x^2 + y^2 + z^2 \leq 1\}$ oriented by the 3-form Ω corresponding to the unit normal

$$\mathbf{N} = \frac{(2x, 2y, 2z, -1)}{\sqrt{4x^2 + 4y^2 + 4z^2 + 1}}.$$

The manifold M is a portion of the 3-manifold given in Example 7 of §8.2 and may be parametrized as

$$\mathbf{X}\colon B \to \mathbf{R}^4, \qquad \mathbf{X}(u_1, u_2, u_3) = (u_1, u_2, u_3, u_1^2 + u_2^2 + u_3^2),$$

where $B = \{(u_1, u_2, u_3) \mid u_1^2 + u_2^2 + u_3^2 \leq 1\}$. Using this parametrization, we have

$$\begin{cases} \mathbf{T}_{u_1} = (1, 0, 0, 2u_1) \\ \mathbf{T}_{u_2} = (0, 1, 0, 2u_2) \\ \mathbf{T}_{u_3} = (0, 0, 1, 2u_3) \\ \mathbf{N} = \dfrac{(2u_1, 2u_2, 2u_3, -1)}{\sqrt{4u_1^2 + 4u_2^2 + 4u_3^2 + 1}} \end{cases},$$

so the orientation 3-form Ω is given by

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \det\begin{bmatrix}\mathbf{N} & \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}.$$

Example 7 of §8.2 shows that the parametrization $\mathbf{X}$ is compatible with this orientation. Hence, we may use this parametrization without any adjustments when we calculate $\int_M d\omega$.

The boundary of M is $\partial M = \{(x, y, z, w) \mid x^2 + y^2 + z^2 = w = 1\}$ and may be parametrized as

$$\mathbf{Y}\colon [0, \pi] \times [0, 2\pi) \to \mathbf{R}^4, \qquad \mathbf{Y}(s_1, s_2) = (\sin s_1 \cos s_2, \sin s_1 \sin s_2, \cos s_1, 1).$$

Then

$$\mathbf{T}_{s_1} = (\cos s_1 \cos s_2, \cos s_1 \sin s_2, -\sin s_1, 0)$$

and

$$\mathbf{T}_{s_2} = (-\sin s_1 \sin s_2, \sin s_1 \cos s_2, 0, 0).$$

[1] For a full and rigorous discussion of differential forms and the generalized Stokes's theorem, see J. R. Munkres, *Analysis on Manifolds* (Addison-Wesley, 1991), Chapters 6 and 7.

An outward-pointing unit vector $\mathbf{V} = (v_1, v_2, v_3, v_4)$ tangent to M and normal to ∂M must satisfy

- $\mathbf{V} \cdot \mathbf{N} = 0$ along ∂M;
- $\mathbf{V} \cdot \mathbf{T}_{s_1} = \mathbf{V} \cdot \mathbf{T}_{s_2} = 0$.

Along ∂M, we have

$$\mathbf{N} = \frac{1}{\sqrt{5}}(2 \sin s_1 \cos s_2, 2 \sin s_1 \sin s_2, 2 \cos s_1, -1).$$

Thus, $\mathbf{V}$ must satisfy the system of equations

$$\begin{cases} (2 \sin s_1 \cos s_2)v_1 + (2 \sin s_1 \sin s_2)v_2 + (2 \cos s_1)v_3 - v_4 = 0 \\ (\cos s_1 \cos s_2)v_1 + (\cos s_1 \sin s_2)v_2 - (\sin s_1)v_3 = 0 \\ -(\sin s_1 \sin s_2)v_1 + (\sin s_1 \cos s_2)v_2 = 0 \end{cases}.$$

After some manipulation, one finds that the unit vector that satisfies these equations and also points away from M is

$$\mathbf{V} = \frac{1}{\sqrt{5}}(\sin s_1 \cos s_2, \sin s_1 \sin s_2, \cos s_1, 2).$$

Then the induced orientation 2-form $\Omega^{\partial M}$ for ∂M is given by

$$\Omega^{\partial M}_{\mathbf{Y}(\mathbf{s})}(\mathbf{a}_1, \mathbf{a}_2) = \Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{V}, \mathbf{a}_1, \mathbf{a}_2),$$

where $\mathbf{X}(\mathbf{u}) = \mathbf{Y}(\mathbf{s})$. In particular, we have

$$\begin{aligned} \Omega^{\partial M}_{\mathbf{Y}(\mathbf{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2}) &= \det\begin{bmatrix} \mathbf{N} & \mathbf{V} & \mathbf{T}_{s_1} & \mathbf{T}_{s_2} \end{bmatrix} \\ &= \det\begin{bmatrix} \frac{2}{\sqrt{5}} \sin s_1 \cos s_2 & \frac{1}{\sqrt{5}} \sin s_1 \cos s_2 & \cos s_1 \cos s_2 & -\sin s_1 \sin s_2 \\ \frac{2}{\sqrt{5}} \sin s_1 \sin s_2 & \frac{1}{\sqrt{5}} \sin s_1 \sin s_2 & \cos s_1 \sin s_2 & \sin s_1 \cos s_2 \\ \frac{2}{\sqrt{5}} \cos s_1 & \frac{1}{\sqrt{5}} \cos s_1 & -\sin s_1 & 0 \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & 0 \end{bmatrix} \\ &= \sin s_1 > 0 \end{aligned}$$

for $0 < s_1 < \pi$. Hence, the parametrization $\mathbf{Y}$ of ∂M, when smooth, is compatible with the induced orientation, so we may use this parametrization to calculate $\int_{\partial M} \omega$.

Now we are ready to integrate. We first compute $\int_M d\omega$. Since $\omega = zw\, dx \wedge dy$, we have

$$\begin{aligned} d\omega &= d(zw) \wedge dx \wedge dy = (z\, dw + w\, dz) \wedge dx \wedge dy \\ &= z\, dw \wedge dx \wedge dy + w\, dz \wedge dx \wedge dy. \end{aligned}$$

Thus,

$$\begin{aligned} \int_M d\omega &= \iiint_B d\omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3})\, du_1\, du_2\, du_3 \\ &= \iiint_B \{u_3\, dw \wedge dx \wedge dy(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}) \\ &\quad + (u_1^2 + u_2^2 + u_3^2)\, dz \wedge dx \wedge dy(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3})\}\, du_1\, du_2\, du_3 \end{aligned}$$

$$= \iiint_B \left\{ u_3 \underbrace{\begin{vmatrix} 2u_1 & 2u_2 & 2u_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}}_{=2u_3} + (u_1^2 + u_2^2 + u_3^2) \underbrace{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}}_{=1} \right\} du_1\, du_2\, du_3$$

$$= \iiint_B (u_1^2 + u_2^2 + 3u_3^2)\, du_1\, du_2\, du_3.$$

Since B is a solid unit ball, the easiest way to evaluate this iterated integral is to use spherical coordinates ρ, φ, and θ. Hence,

$$\begin{aligned}
\int_M d\omega &= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \left(\rho^2 + 2\rho^2 \cos^2 \varphi\right) \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \left(\sin\varphi + 2\cos^2\varphi \sin\varphi\right) d\rho\, d\varphi\, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi} \frac{1}{5} \left(\sin\varphi + 2\cos^2\varphi \sin\varphi\right) d\varphi\, d\theta \\
&= \frac{1}{5} \int_0^{2\pi} \left(-\cos\varphi - \frac{2}{3}\cos^3\varphi\right)\Big|_{\varphi=0}^{\pi} d\theta \\
&= \frac{1}{5} \int_0^{2\pi} \frac{10}{3}\, d\theta = \frac{4\pi}{3}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\partial M} \omega &= \iint_{[0,\pi]\times[0,2\pi)} \omega_{\mathbf{Y}(\mathbf{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2})\, ds_1\, ds_2 \\
&= \int_0^{2\pi} \int_0^{\pi} \cos s_1\, dx \wedge dy(\mathbf{T}_{s_1}, \mathbf{T}_{s_2})\, ds_1\, ds_2 \\
&= \int_0^{2\pi} \int_0^{\pi} \cos s_1 \begin{vmatrix} \cos s_1 \cos s_2 & -\sin s_1 \sin s_2 \\ \cos s_1 \sin s_2 & \sin s_1 \cos s_2 \end{vmatrix} ds_1\, ds_2 \\
&= \int_0^{2\pi} \int_0^{\pi} \cos s_1 \left(\cos s_1 \sin s_1\right) ds_1\, ds_2 \\
&= \int_0^{2\pi} \int_0^{\pi} \cos^2 s_1 \sin s_1\, ds_1\, ds_2 \\
&= \int_0^{2\pi} \left(-\frac{1}{3}\cos^3 s_1\right)\Big|_{s_1=0}^{\pi} ds_2 = \int_0^{2\pi} \frac{2}{3}\, ds_2 = \frac{4\pi}{3}.
\end{aligned}$$

Therefore, the generalized Stokes's theorem is verified in this case. ◆

Besides being notationally elegant, the integral formula in Theorem 3.2 beautifully encompasses all three of the major results of vector analysis, as we now show.

First, let ω be a 1-form defined on an open set U in $\mathbf{R}^2$. Then

$$\omega = M(x, y)\,dx + N(x, y)\,dy,$$

so that

$$\begin{aligned} d\omega &= dM \wedge dx + dN \wedge dy \\ &= \left(\frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy\right) \wedge dx + \left(\frac{\partial N}{\partial x}dx + \frac{\partial N}{\partial y}dy\right) \wedge dy \\ &= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge dy. \end{aligned}$$

The generalized Stokes's theorem (Theorem 3.2) says that if D is a 2-manifold contained in U and ∂D is given the induced orientation (see Figure 8.9), then

$$\int_D d\omega = \int_{\partial D} \omega,$$

or, in this instance, that

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_{\partial D} M dx + N dy,$$

which is Green's theorem.

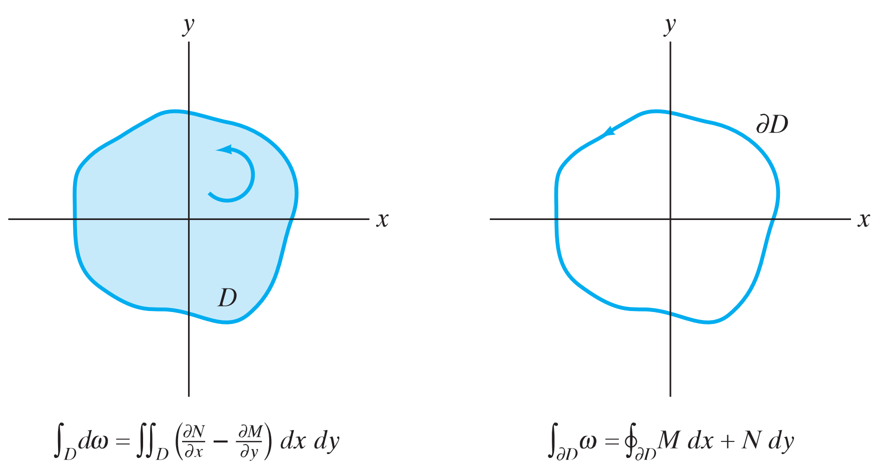

Figure 8.9 The generalized Stokes's theorem implies Green's theorem.

Next, suppose ω is a 1-form defined on an open set U in $\mathbf{R}^3$. Then

$$\omega = F_1(x, y, z)\,dx + F_2(x, y, z)\,dy + F_3(x, y, z)\,dz.$$

It follows that

$$\begin{aligned} d\omega = &\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx \\ &+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy. \end{aligned}$$

Recall from Proposition 2.2 that if S is a parametrized 2-manifold (surface in $\mathbf{R}^3$), then

$$\int_{\partial S} \omega = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s},$$

where $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$. From Proposition 2.4,

$$\int_S d\omega = \iint_S \mathbf{G} \cdot d\mathbf{S},$$

where

$$\mathbf{G} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} = \nabla \times \mathbf{F}.$$

Theorem 3.2 tells us, if S is oriented and ∂S is given the induced orientation, that

$$\int_{\partial S} \omega = \int_S d\omega,$$

or, equivalently, that

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

which is the classical Stokes's theorem. (See Figure 8.10.)

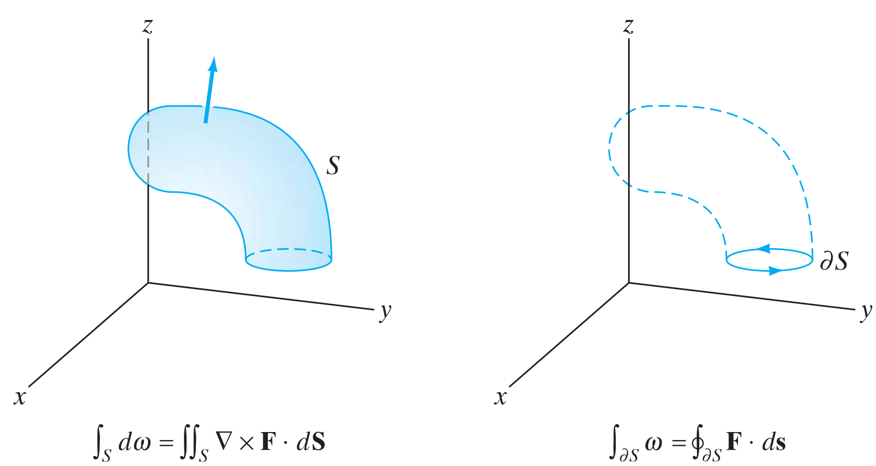

Figure 8.10 The generalized Stokes's theorem gives the classical Stokes's theorem.

Finally, let ω be a 2-form defined on an open set in $\mathbf{R}^3$. So

$$\omega = F_1(x, y, z)\,dy \wedge dz + F_2(x, y, z)\,dz \wedge dx + F_3(x, y, z)\,dx \wedge dy.$$

You can check that

$$d\omega = \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz.$$

If D is a region in $\mathbf{R}^3$, then D is automatically a parametrized 3-manifold, since the map $\mathbf{X}\colon D \to \mathbf{R}^3$, $\mathbf{X}(x, y, z) = (x, y, z)$ parametrizes D. (One can show that in this instance D is always orientable as well.) If D is bounded and ∂D (which is a surface) is given the induced orientation (i.e., outward-pointing normal), then

Proposition 2.4 states that

$$\int_{\partial D} \omega = \oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$. From Example 6 of §8.2,

$$\int_D d\omega = \int_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx \wedge dy \wedge dz = \iiint_D \nabla \cdot \mathbf{F}\, dV.$$

Theorem 3.2 indicates that $\int_{\partial D} \omega = \int_D d\omega$ or

$$\oiint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV,$$

which is, of course, Gauss's theorem. (See Figure 8.11.)

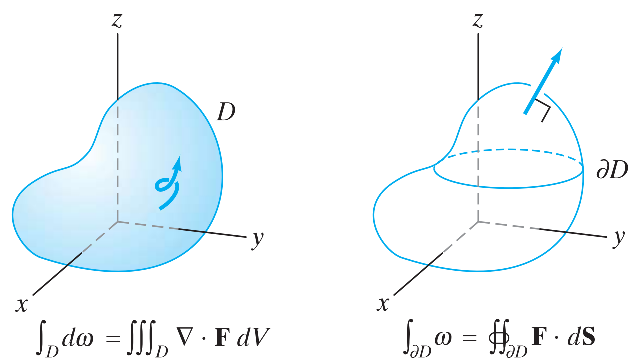

Figure 8.11 The generalized Stokes's theorem gives rise to Gauss's theorem.

In the foregoing remarks, we have implicitly set up a sort of "dictionary" between the language of differential forms and exterior derivatives and that of scalar and vector fields. To be explicit, see the table of correspondences shown in Figure 8.12.

The theorems of Green, Stokes, and Gauss all arise from Theorem 3.2 applied to 1-forms and 2-forms. The next question is, can the "dictionary" and Theorem 3.2 provide a corresponding result for 0-forms? The generalized Stokes's theorem (Theorem 3.2) states, for a 0-form ω and an oriented parametrized curve C, that

$$\int_C d\omega = \int_{\partial C} \omega.$$

k	Differential k-form	Field	Derivative
0	ω	Scalar field f	$d\omega \leftrightarrow \nabla f$
1	$\omega = F_1\,dx + F_2\,dy + F_3\,dz$	Vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$	$d\omega \leftrightarrow \nabla \times \mathbf{F}$
2	$\omega = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$	Vector field $\mathbf{F} = F_1\,\mathbf{i} + F_2\,\mathbf{j} + F_3\,\mathbf{k}$	$d\omega \leftrightarrow \nabla \cdot \mathbf{F}$

Figure 8.12 A differential forms–vector fields dictionary.

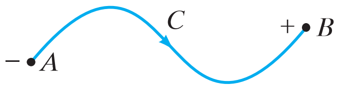

Figure 8.13 The orientation of the curve C induces an orientation of its boundary (i.e., the endpoints A and B).

Now, if C is closed, then ∂C is empty (and so $\int_{\partial C} \omega = 0$). But if C is not closed, then ∂C consists of just two points. In that case, what should $\int_{\partial C} \omega$ mean? In particular, to apply Theorem 3.2, we must orient ∂C in a manner that is consistent with the orientation of C, which can be done by assigning a "−" sign to the initial point A of C and a "+" sign to the terminal point B. (See Figure 8.13.) Then $\int_{\partial C} \omega$ is just $f(B) - f(A)$, where f is the function (scalar field) corresponding to ω in the table. Since $d\omega$ corresponds to ∇f, Theorem 3.2 tells us that

$$\int_C \nabla f \cdot d\mathbf{s} = f(B) - f(A), \tag{1}$$

the result of Theorem 3.3 in Chapter 6.

Finally, for the case $n = 1$, that is, the case of 0-forms (functions) on $\mathbf{R}$, the 0-form ω corresponds to a function f of a single variable, and ∇f is the ordinary derivative f'. Furthermore, a parametrized curve in $\mathbf{R}$ is simply a closed interval $[a, b]$. Then equation (1) reduces to

$$\int_a^b f'(x)\, dx = f(b) - f(a),$$

a version of the fundamental theorem of calculus. Thus, we can appreciate that the generalized Stokes's theorem is an elegant and powerful generalization of the fundamental theorem of calculus to arbitrary dimensions.

8.3 Exercises

In Exercises 1–7, determine $d\omega$, where ω is as indicated.

1. $\omega = e^{xyz}$

2. $\omega = x^3 y - 2xz^2 + xy^2 z$

3. $\omega = (x^2 + y^2)\, dx + xy\, dy$

4. $\omega = x_1\, dx_2 - x_2\, dx_1 + x_3 x_4\, dx_4 - x_4 x_5\, dx_5$

5. $\omega = xz\, dx \wedge dy - y^2 z\, dx \wedge dz$

6. $\omega = x_1 x_2 x_3\, dx_2 \wedge dx_3 \wedge dx_4 + x_2 x_3 x_4\, dx_1 \wedge dx_2 \wedge dx_3$

7. $\omega = \sum_{i=1}^n x_i^2\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ (Note: $\widehat{dx_i}$ means that the term dx_i is omitted.)

8. Let $\mathbf{u}$ be a unit vector and f a differentiable function. Show that $df_{\mathbf{x}_0}(\mathbf{u}) = D_{\mathbf{u}} f(\mathbf{x}_0)$. (Recall that $D_{\mathbf{u}} f(\mathbf{x}_0)$ denotes the directional derivative of f at $\mathbf{x}_0$ in the direction of $\mathbf{u}$.)

9. If $\omega = F(x, z)\, dy + G(x, y)\, dz$ is a (differentiable) 1-form on $\mathbf{R}^3$, what can F and G be so that $d\omega = z\, dx \wedge dy + y\, dx \wedge dz$?

10. Verify the generalized Stokes's theorem (Theorem 3.2) for the 3-manifold M of Exercise 11 of §8.2, where $\omega = 2x\, dy \wedge dz - z\, dx \wedge dy$.

11. Verify the generalized Stokes's theorem (Theorem 3.2) for the 3-manifold

$$M = \{(x, y, z, w) \in \mathbf{R}^4 \mid x = 8 - 2y^2 - 2z^2 - 2w^2, x \geq 0\}$$

and the 2-form $\omega = xy\, dz \wedge dw$. (Hint: First compute $\int_{\partial M} \omega$. To calculate $\int_M d\omega$, study Example 3 of this section.)

12. (a) Let M be a parametrized 3-manifold in $\mathbf{R}^3$ (i.e., a solid). Show that

$$\text{Volume of } M = \frac{1}{3} \int_{\partial M} x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy.$$

(b) Let M be a parametrized n-manifold in $\mathbf{R}^n$. Explain why we should have

$$n\text{-dimensional volume of } M = \frac{1}{n} \int_{\partial M} x_1\, dx_2 \wedge \cdots \wedge dx_n - x_2\, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + x_3\, dx_1 \wedge dx_2 \wedge dx_4 \wedge \cdots \wedge dx_n + \cdots + (-1)^{n-1} x_n\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n-1}.$$

True/False Exercises for Chapter 8

1. $(dx \wedge dy + dy \wedge dz)((1, 0, 1), (0, -1, 3)) = 0$.
2. $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = dx_2 \wedge dx_4 \wedge dx_1 \wedge dx_3$.
3. There are 21 basic 5-forms in $\mathbf{R}^7$.
4. $dx_1 \wedge dx_2 = dx_2 \wedge dx_1$.
5. $(dx_1 \wedge dx_2) \wedge dx_3 = dx_3 \wedge (dx_1 \wedge dx_2)$.
6. If ω is a 3-form on $\mathbf{R}^6$ and η is a 5-form on $\mathbf{R}^6$, then $\omega \wedge \eta = \eta \wedge \omega$.
7. If ω is a 2-form on $\mathbf{R}^8$ and η is a 3-form on $\mathbf{R}^8$, then $\omega \wedge \eta = \eta \wedge \omega$.
8. $dx \wedge dy \wedge dz(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -dz \wedge dy \wedge dx(\mathbf{a}, \mathbf{c}, \mathbf{b})$.
9. $dx_i \wedge dx_j(\mathbf{a}, \mathbf{b}) = -dx_i \wedge dx_j(\mathbf{b}, \mathbf{a})$.
10. Let $D = [0, 2] \times [-1, 1]$ and let $\mathbf{X}: D \to \mathbf{R}^4$ be given by
$$\mathbf{X}(s, t) = (s - t,\ st^2,\ se^t, 4t).$$
Then $M = \mathbf{X}(D)$ is a smooth parametrized 2-manifold in $\mathbf{R}^4$.
11. Let $D = [-2, 2] \times [0, 5] \times [-3, 3]$ and let $\mathbf{X}: D \to \mathbf{R}^4$ be given by
$$\mathbf{X}(u_1, u_2, u_3) = (u_1 u_3^2,\ u_2^2 \cos u_3,\ u_1 - u_2,\ u_2^3 u_3^4).$$
Then $M = \mathbf{X}(D)$ is a smooth parametrized 3-manifold in $\mathbf{R}^4$.
12. If $D = [0, 1] \times [0, 1]$, then the underlying manifolds of $\mathbf{X}: D \to \mathbf{R}^3$,
$$\mathbf{X}(s, t) = (s \cos 2\pi t,\ s \sin 2\pi t,\ s^2)$$
and $\mathbf{Y}: D \to \mathbf{R}^3$,
$$\mathbf{Y}(s, t) = (t \cos 2\pi s,\ t \sin 2\pi s,\ t^2)$$
are the same.
13. Let $\omega = dx \wedge dy$ and $D = [0, 1] \times [0, 1]$. Then $\int_{\mathbf{X}} \omega = \int_{\mathbf{Y}} \omega$, where $\mathbf{X}: D \to \mathbf{R}^3$,
$$\mathbf{X}(s, t) = (s \cos 2\pi t,\ s \sin 2\pi t,\ s^2),$$
and $\mathbf{Y}: D \to \mathbf{R}^3$,
$$\mathbf{Y}(s, t) = (t \cos 2\pi s,\ t \sin 2\pi s,\ t^2).$$
14. Let $B = \{\mathbf{u} \in \mathbf{R}^3 \mid u_1^2 + u_2^2 + u_3^2 \leq 1\}$. The generalized paraboloid $\mathbf{X}: B \to \mathbf{R}^4$ defined by
$$\mathbf{X}(u_1, u_2, u_3) = (u_1, u_2, u_3, u_1^2 + 2u_2^2 + 3u_3^2)$$
has as its boundary the ellipsoid $\mathbf{Y}: [0, \pi] \times [0, 2\pi) \to \mathbf{R}^4$,
$$\mathbf{Y}(s, t) = \left(\sin s \cos t,\ \tfrac{1}{\sqrt{2}} \sin s \sin t,\ \tfrac{1}{\sqrt{3}} \cos s,\ 1\right).$$
15. Let $M \subseteq \mathbf{R}^n$ be the graph of a function $f: U \subseteq \mathbf{R}^{n-1} \to \mathbf{R}^n$ parametrized by $\mathbf{X}: U \to \mathbf{R}^n$,
$$\mathbf{X}(u_1, \ldots, u_{n-1}) = (u_1, \ldots, u_{n-1}, f(u_1, \ldots, u_{n-1})).$$
If
$$\mathbf{N}(u_1, \ldots, u_{n-1}) = \frac{(f_{u_1}, \ldots, f_{u_{n-1}}, -1)}{\sqrt{f_{u_1}^2 + \cdots + f_{u_{n-1}}^2 + 1}}$$
is a unit normal, then the parametrization $\mathbf{X}$ is compatible with the $(n-1)$-form Ω defined by
$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}) = \det\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{n-1} & \mathbf{N} \end{bmatrix}.$$
16. If $\omega = x_1 x_3\, dx_2 \wedge dx_4$, then $d\omega = x_3\, dx_1 \wedge dx_2 \wedge dx_4 + x_1\, dx_2 \wedge dx_3 \wedge dx_4$.
17. If $\omega = x_1\, dx_3 - x_2\, dx_1 + x_1 x_2 x_3\, dx_3$, then
$$d\omega = (x_2 x_3 + 1)\, dx_1 \wedge dx_3 + dx_1 \wedge dx_2 + x_1 x_3\, dx_2 \wedge dx_3.$$
18. If $\omega = x_1 x_2\, dx_1 \wedge dx_2 + x_2 x_3\, dx_1 \wedge dx_3 + x_1 x_3\, dx_2 \wedge dx_3$, then
$$d\omega = 2x_3\, dx_1 \wedge dx_2 \wedge dx_3.$$
19. If ω is an n-form on $\mathbf{R}^n$, then $d\omega = 0$.
20. If M is a parametrized k-manifold without boundary in $\mathbf{R}^n$ and ω is $(k-1)$-form defined on an open set containing M, then $\int_M d\omega = 0$.

Miscellaneous Exercises for Chapter 8

1. Let ω be a k-form, η an l-form. Show that
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$
This is accomplished by the following steps:
(a) Show that the result is true when $k = l = 0$, that is, when $\omega = f$ and $\eta = g$. (Here f and g are scalar-valued functions.)
(b) Establish the result when $k = 0$ and $l > 0$.
(c) Establish the result when $k > 0$ and $l = 0$.
(d) Establish the result when k and l are both positive.
2. Let M be the subset of $\mathbf{R}^5$ described as $\{(x_1, x_2, x_3, x_4, x_5) \mid x_5 = x_1 x_2 x_3 x_4,\ \ 0 \leq x_1, x_2, x_3, x_4 \leq 1\}$.

(a) Give a parametrization for M (as a 4-manifold) and check that your parametrization is compatible with the orientation 4-form $\Omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.

(b) Calculate $\int_M x_4\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$.

3. (a) Let C be the curve in $\mathbf{R}^2$ given by $y = f(x)$, $a \le x \le b$. Assume that f is of class C^1. If C is oriented by the direction in which x increases, show that if $\omega = y\, dx$, then

$$\int_C \omega = \text{area under the graph of } f.$$

(b) Let S be the surface in $\mathbf{R}^3$ given by the equation $z = f(x, y)$, where $(x, y) \in [a, b] \times [c, d]$. Assume that f is of class C^1. If S is oriented by upward-pointing normal, show that if $\omega = z\, dx \wedge dy$, then

$$\int_S \omega = \text{volume under the graph of } f.$$

(c) Now we generalize parts (a) and (b) as follows: Suppose $f\colon D \to \mathbf{R}$ is a function of class C^1 defined on a connected region $D \subseteq \mathbf{R}^{n-1}$. Let M be the $(n-1)$-dimensional hypersurface in $\mathbf{R}^n$ defined by the equation $x_n = f(x_1, \ldots, x_{n-1})$, where $(x_1, \ldots, x_{n-1}) \in D$. If $\omega = x_n\, dx_1 \wedge \cdots \wedge dx_{n-1}$, show that

$$\int_M \omega = \pm(n\text{-dimensional volume under the graph of } f).$$

How can we guarantee a "+" sign in the equation?

4. Let M be the portion of the cylinder $x^2 + z^2 = 1$, $0 \le y \le 3$, oriented by unit normal $\mathbf{N} = (x, 0, z)$.

(a) Use $\mathbf{N}$ to give an orientation 2-form Ω for M. Find a parametrization for M compatible with Ω.

(b) Identify ∂M and parametrize it.

(c) Determine the orientation form $\Omega^{\partial M}$ for ∂M induced from Ω of part (a).

(d) Verify the generalized Stokes's theorem (Theorem 3.2) for M and $\omega = z\, dx + (x + y + z)\, dy - x\, dz$.

5. Use the generalized Stokes's theorem to calculate $\int_{S^4} \omega$, where S^4 denotes the unit 4-sphere $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 1\}$ and $\omega = x_3\, dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 + x_4\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$.

6. (a) Let ω be a 0-form (i.e., a function) of class C^2. Show that $d(d\omega) = 0$.

(b) Now suppose that ω is a k-form of class C^2, meaning that when ω is written as

$$\sum_{1 \le i_1 < \cdots < i_k \le n} F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

each $F_{i_1 \ldots i_k}$ is of class C^2. Use part (a) and the result of Exercise 1 to show that $d(d\omega) = 0$.

7. In this problem, show that the equation $d(d\omega) = 0$ implies two well-known results about scalar and vector fields.

(a) First, let ω be a 0-form (of class C^2). Then ω corresponds to a scalar field f. Use the chart on page 559 to interpret the equation $d(d\omega) = 0$.

(b) Next, suppose that ω is a 1-form (again of class C^2). Then ω corresponds to a vector field. Interpret the equation $d(d\omega) = 0$ in this case.

8. Let

$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

(a) Evaluate $\int_S \omega$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by outward normal.

(b) Calculate $d\omega$.

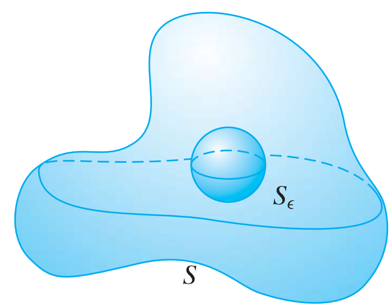

Figure 8.14 Figure for Exercise 8.

(c) Verify Theorem 3.2 over the region $M = \{(x, y, z) \mid a^2 \le x^2 + y^2 + z^2 \le 1\}$, where $a \ne 0$.

(d) Now let M be the solid unit ball $x^2 + y^2 + z^2 \le 1$. Does Theorem 3.2 hold for M and ω? Why or why not?

(e) Suppose that S is any closed, bounded surface that lies entirely outside the sphere $S_\epsilon = \{(x, y, z) \mid x^2 + y^2 + z^2 = \epsilon^2\}$. (See Figure 8.14.) Argue that if S is oriented by outward normal, then $\int_S \omega = 4\pi$.

9. Let M be an oriented $(k + l + 1)$-manifold in $\mathbf{R}^n$; let ω be a k-form and η an l-form defined on an open set of $\mathbf{R}^n$ that contains M. If $\partial M = \varnothing$, use Theorem 3.2 and Exercise 1 to show that

$$\int_M d\omega \wedge \eta = (-1)^{k+1} \int_M \omega \wedge d\eta.$$

10. Let M be an oriented k-manifold. Use Exercise 1 and the general version of Stokes's theorem to establish "integration by parts" for k-forms ω and 0-forms f:

$$\int_M f\, d\omega = \int_{\partial M} f\omega - \int_M df \wedge \omega.$$