7.4 Further Vector Analysis; Maxwell's Equations

In this section, we use Gauss's theorem and Stokes's theorem first to prove some abstract results in vector analysis and then to study Maxwell's equations of electricity and magnetism.

Green's Formulas

Our purpose in §7.4 is to establish a few fundamental results of vector analysis. Throughout the discussion all scalar and vector fields are defined on subsets of $\mathbf{R}^3$.

The following pair of results is established readily:

THEOREM 4.1 (GREEN'S FIRST AND SECOND FORMULAS) Let f and g be scalar fields of class C^2, and let D be a solid region in space, bounded by a piecewise smooth surface $S = \partial D$, oriented as in Gauss's theorem. Then we have

Green's first formula:

$$\iiint_D \nabla f \cdot \nabla g \, dV + \iiint_D f \, \nabla^2 g \, dV = \oiint_S f \, \nabla g \cdot d\mathbf{S}.$$

Green's second formula:

$$\iiint_D \left(f \, \nabla^2 g - g \, \nabla^2 f \right) dV = \oiint_S (f \, \nabla g - g \, \nabla f) \cdot d\mathbf{S}.$$

PROOF The first formula follows from Gauss's theorem applied to the vector field $\mathbf{F} = f \nabla g$. (We leave the details to you.) The second formula follows from writing the first formula twice:

$$\iiint_D \nabla f \cdot \nabla g \, dV + \iiint_D f \nabla^2 g \, dV = \oiint_S f \nabla g \cdot d\mathbf{S}; \tag{1}$$

$$\iiint_D \nabla g \cdot \nabla f \, dV + \iiint_D g \nabla^2 f \, dV = \oiint_S g \nabla f \cdot d\mathbf{S}. \tag{2}$$

Now, subtract equation (2) from equation (1). ■

The third of Green's formulas requires considerably more work to prove.

THEOREM 4.2 (GREEN'S THIRD FORMULA) Assume f is a function of class C^2. Then, for $\partial D = S$ oriented as in Gauss's theorem and points $\mathbf{r}$ in the interior of D,

$$f(\mathbf{r}) = -\frac{1}{4\pi} \iiint_D \frac{\nabla^2 f(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|} \, dV + \frac{1}{4\pi} \oiint_S \left(-f(\mathbf{x}) \nabla \left(\frac{1}{\|\mathbf{r} - \mathbf{x}\|} \right) + \frac{\nabla f(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|} \right) \cdot d\mathbf{S}.$$

In this formula, dV denotes integration with respect to the variables in $\mathbf{x} = (x, y, z)$ (i.e., $\mathbf{r} = (r_1, r_2, r_3)$ is fixed in the integration), and the symbol ∇ means $\nabla_{\mathbf{x}}$, differentiation with respect to x, y, and z.

A proof of Theorem 4.2 appears in the addendum to this section.

An Inversion Formula for the Laplacian

Green's third formula is a type of **inversion formula**—a formula that enables us to recover the values of a function f by knowing certain integrals involving its gradient and Laplacian. Green's third formula is mainly of technical interest in proving further results. We use it here to obtain an inversion formula for the Laplacian operator.

We begin by applying the Laplacian $\nabla_{\mathbf{r}}^2$ to Green's third formula:

$$\nabla_{\mathbf{r}}^2 f(\mathbf{r}) = \nabla_{\mathbf{r}}^2 \left[-\frac{1}{4\pi} \iiint_D \frac{\nabla_{\mathbf{x}}^2 f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\, dV + \frac{1}{4\pi} \oiint_S \left(\frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|} - f(\mathbf{x}) \nabla_{\mathbf{x}} \left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|} \right) \right) \cdot d\mathbf{S} \right] \tag{3}$$

The trick is to move $\nabla_{\mathbf{r}}^2$ inside the surface integral, which is justified since $\mathbf{x} \neq \mathbf{r}$ when $\mathbf{x}$ varies over S:

$$\nabla_{\mathbf{r}}^2 \oiint_S \left(\frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|} - f(\mathbf{x}) \nabla_{\mathbf{x}} \left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|} \right) \right) \cdot d\mathbf{S} = \oiint_S \nabla_{\mathbf{r}}^2 \left(\frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|} - f(\mathbf{x}) \nabla_{\mathbf{x}} \left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|} \right) \right) \cdot d\mathbf{S}.$$

By direct calculation, $\nabla_{\mathbf{r}}^2(1/\|\mathbf{r}-\mathbf{x}\|) = 0$ for $\mathbf{x} \neq \mathbf{r}$, so

$$\nabla_{\mathbf{r}}^2 \left(\frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|} \right) = \mathbf{0}.$$

Similarly, since $f(\mathbf{x})$ does not involve $\mathbf{r}$,

$$\begin{aligned} \nabla_{\mathbf{r}}^2 \left(f(\mathbf{x}) \nabla_{\mathbf{x}} \left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|} \right) \right) &= f(\mathbf{x}) \nabla_{\mathbf{r}}^2 \nabla_{\mathbf{x}} \left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|} \right) \\ &= f(\mathbf{x}) \nabla_{\mathbf{x}} \nabla_{\mathbf{r}}^2 \left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|} \right) \\ &= \mathbf{0}. \end{aligned}$$

Therefore, the Laplacian of the original surface integral is 0. We may conclude from equation (3) that, for $\mathbf{r}$ in the interior of D,

$$\nabla_{\mathbf{r}}^2 f(\mathbf{r}) = -\frac{1}{4\pi} \nabla_{\mathbf{r}}^2 \iiint_D \frac{\nabla_{\mathbf{x}}^2 f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\, dV,$$

and we have shown the following:

THEOREM 4.3 If $\varphi = \nabla^2 f$ for some function f of class C^2, then for $\mathbf{r}$ in the interior of D,

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi} \nabla_{\mathbf{r}}^2 \iiint_D \frac{\varphi(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\, dV.$$

Theorem 4.3 provides an inversion formula for the Laplacian in the following sense: If $\nabla^2 f = \varphi$, then

$$f(\mathbf{r}) = -\frac{1}{4\pi} \iiint_D \frac{\varphi(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\, dV + g(\mathbf{r}),$$

where g is any harmonic function (i.e., g is such that $\nabla^2 g = 0$). That is, if the Laplacian of f is known, we can recover the function f itself, up to addition of a harmonic function.

Finally, it can be shown that the result of Theorem 4.3 holds under considerably less stringent hypotheses than having φ be the Laplacian of another function.[1]

Maxwell's Equations

Maxwell's equations are fundamental results that govern the behavior of—and interactions between—electric and magnetic fields. We see how Maxwell's equations arise from a few simple physical principles coupled with the vector analysis discussed previously.

Gauss's law for electric fields If $\mathbf{E}$ is an electric field, then the flux of $\mathbf{E}$ across a closed surface S is

$$\text{Flux of } \mathbf{E} = \oiint_S \mathbf{E} \cdot d\mathbf{S}. \tag{4}$$

Applying Gauss's theorem to formula (4), we find that

$$\text{Flux of } \mathbf{E} = \iiint_D \nabla \cdot \mathbf{E}\, dV, \tag{5}$$

where D is the region enclosed by S.

If the electric field $\mathbf{E}$ is determined by a single point charge of q coulombs located at the origin, then $\mathbf{E}$ is given by

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{x}}{\|\mathbf{x}\|^3}, \tag{6}$$

where $\mathbf{x} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. In mks units, $\mathbf{E}$ is measured in volts/meter. The constant ϵ_0 is known as the **permittivity of free space**; its value (in mks units) is

$$8.854 \times 10^{-12} \text{ coulomb}^2/\text{newton-meter}^2.$$

For the electric field in equation (6), we can readily verify that $\nabla \cdot \mathbf{E} = 0$ wherever $\mathbf{E}$ is defined. From formulas (4) and (5), if S is any surface that does *not* enclose the origin, then the flux of $\mathbf{E}$ across S is zero.

But now a question arises: How do we calculate the flux of the electric field in equation (6) across surfaces that *do* enclose the origin? The trick is to find an appropriate way to exclude the origin from consideration. To that end, first suppose that S_b denotes a sphere of radius b centered at the origin (i.e., S_b has equation $x^2 + y^2 + z^2 = b^2$). Then the outward unit normal to S_b is

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{b} = \frac{1}{b}\mathbf{x}.$$

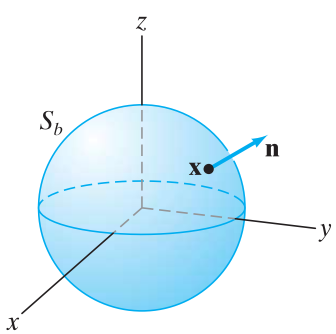

Figure 7.53 The sphere S_b of radius b, centered at the origin, together with an outward unit normal vector.

(See Figure 7.53.) From equation (6),

$$\oiint_{S_b} \mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\epsilon_0} \oiint_{S_b} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot \frac{1}{b}\mathbf{x}\, dS.$$

On S_b, we have $\|\mathbf{x}\| = b$, so that

$$\begin{aligned}
\oiint_{S_b} \mathbf{E} \cdot d\mathbf{S} &= \frac{q}{4\pi\epsilon_0} \oiint_{S_b} \frac{\mathbf{x}}{b^3} \cdot \frac{\mathbf{x}}{b}\, dS = \frac{q}{4\pi\epsilon_0} \oiint_{S_b} \frac{\|\mathbf{x}\|^2}{b^4}\, dS \\
&= \frac{q}{4\pi\epsilon_0} \oiint_{S_b} \frac{b^2}{b^4}\, dS = \frac{q}{4\pi\epsilon_0 b^2} \oiint_{S_b} dS \\
&= \frac{q}{4\pi\epsilon_0 b^2} (\text{surface area of } S_b) \\
&= \frac{q}{4\pi\epsilon_0 b^2} (4\pi b^2) = \frac{q}{\epsilon_0}.
\end{aligned}$$

[1] See, for example, O. D. Kellogg, *Foundations of Potential Theory* (Springer, 1928; reprinted by Dover Publications, 1954), p. 220.

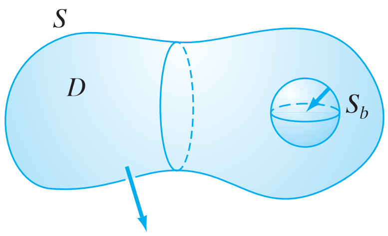

Figure 7.54 The solid region D is the region inside S and outside S_b.

Now, suppose S is *any* surface enclosing the origin. Let S_b be a small sphere centered at the origin and contained inside S. Let D be the solid region in $\mathbf{R}^3$ between S_b and S. (See Figure 7.54.) Note that $\nabla \cdot \mathbf{E} = 0$ throughout D, since D does not contain the origin. ($\mathbf{E}$ is still defined as in equation (6).) Orienting $\partial D = S \cup S_b$ with normals that point away from D, we obtain

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} - \oiint_{S_b} \mathbf{E} \cdot d\mathbf{S} = \oiint_{\partial D} \mathbf{E} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{E}\, dV = 0,$$

using Gauss's theorem. We conclude that

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0} \tag{7}$$

for any surface that encloses the origin. By modifying equation (6) for $\mathbf{E}$ appropriately, we can show that formula (7) also holds for any closed surface containing a single point charge of q coulombs located anywhere. (Note: The arguments just given hold for *any* inverse square law vector field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}/\|\mathbf{x}\|^3$, where k is a constant. See Exercise 13 in this section for details.)

We can adapt the arguments just given to accommodate the case of n discrete point charges. For $i = 1, \ldots, n$, suppose a point charge of q_i coulombs is located at position $\mathbf{r}_i$. The electric field $\mathbf{E}$ for this configuration is

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} q_i \frac{\mathbf{x} - \mathbf{r}_i}{\|\mathbf{x} - \mathbf{r}_i\|^3}. \tag{8}$$

For $\mathbf{E}$ as given in equation (8), we can calculate that $\nabla \cdot \mathbf{E} = 0$, except at $\mathbf{x} = \mathbf{r}_i$. If S is any closed, piecewise smooth, outwardly-oriented surface containing the charges, then we may use Gauss's theorem to find the flux of $\mathbf{E}$ across S by taking n small spheres $S_1, S_2, \ldots, S_n$, each enclosing a single point charge. (See Figure 7.55.) If D is the region inside S but outside all the spheres, we have, by choosing appropriate orientations and using Gauss's theorem,

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} - \sum_{i=1}^{n} \oiint_{S_i} \mathbf{E} \cdot d\mathbf{S} = \oiint_{\partial D} \mathbf{E} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{E}\, dV = 0,$$

since $\nabla \cdot \mathbf{E} = 0$ on D. Hence,

$$\oiint_S \mathbf{E} \cdot d\mathbf{S} = \sum_{i=1}^{n} \oiint_{S_i} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \sum_{i=1}^{n} q_i$$

$$= \frac{1}{\epsilon_0} \text{(total charge enclosed by } S\text{)}. \tag{9}$$

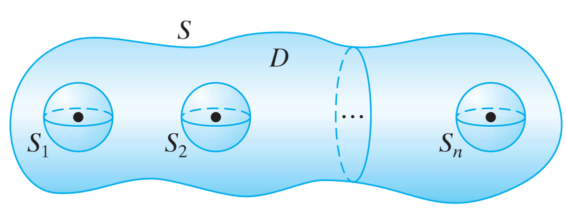

Figure 7.55 D is the solid region inside the surface S and outside the small spheres $S_1, S_2, \ldots, S_n$, each enclosing a point charge.

To establish Gauss's law, consider the case not of an electric field determined by discrete point charges, but rather of one determined by a **continuous charge distribution** given by a **charge density** $\rho(\mathbf{x})$. The total charge over a region D in space is

$$\iiint_D \rho(\mathbf{x})\, dV,$$

so that, in place of formula (8), we have an electric field,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\iiint_D \rho(\mathbf{x})\frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\, dV. \tag{10}$$

In equation (10), the integration occurs with respect to the variables in $\mathbf{x}$. (Note: It is not at all obvious that the integral used to define $\mathbf{E}(\mathbf{r})$ converges at points $\mathbf{r} \in D$, where $\rho(\mathbf{r}) \neq 0$, because at such points the triple integral in equation (10) is improper. See Exercise 20 in this section for an indication of how to deal with this issue.)

The integral form of Gauss's law, analogous to that of formula (9), is

$$\oiint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\epsilon_0}\iiint_D \rho\, dV, \tag{11}$$

where $S = \partial D$. If we apply Gauss's theorem to the left side of formula (11), we find that

$$\iiint_D \nabla\cdot\mathbf{E}\, dV = \frac{1}{\epsilon_0}\iiint_D \rho\, dV.$$

Since the region D is arbitrary, it may be "shrunk to a point." From this, we conclude that

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}. \tag{12}$$

Equation (12) is the differential form of Gauss's law.

Magnetic fields A moving charged particle generates a magnetic field. To be specific, if a point charge of q coulombs is at position $\mathbf{r}_0$ and is moving with velocity $\mathbf{v}$, then the magnetic field it induces is

$$\mathbf{B}(\mathbf{r}) = \left(\frac{\mu_0 q}{4\pi}\right)\left(\frac{\mathbf{v}\times(\mathbf{r}-\mathbf{r}_0)}{\|\mathbf{r}-\mathbf{r}_0\|^3}\right). \tag{13}$$

In mks units, $\mathbf{B}$ is measured in teslas. The constant μ_0 is known as the **permeability of free space**; in mks units

$$\mu_0 = 4\pi \times 10^{-7}\ \text{N/amp}^2 \approx 1.257 \times 10^{-6}\ \text{N/amp}^2.$$

In the case of a magnetic field that arises from a continuous, charged medium (such as electric current moving through a wire), rather than from a single moving charge, we replace q by a suitable charge density function ρ and the velocity of a single particle by the **velocity vector field v** of the charges. Then we define the **current density field J** by

$$\mathbf{J}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{v}(\mathbf{x}). \tag{14}$$

In place of formula (13), we use the following definition for the magnetic field resulting from moving charges in a region D in space:

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\iiint_D \rho(\mathbf{x})\mathbf{v}(\mathbf{x}) \times \frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV \\ &= \frac{\mu_0}{4\pi}\iiint_D \mathbf{J}(\mathbf{x}) \times \frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV. \qquad (15)\end{aligned}$$

In equation (15), as in equation (10), the integration is with respect to the variables constituting $\mathbf{x}$. As in equation (10), it is not obvious that the integrals in equation (15) are convergent if $\mathbf{r} \in D$, but, in fact, they are. (See Exercise 21 in this section.)

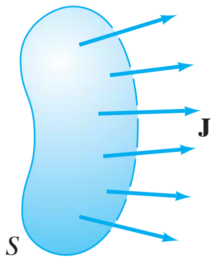

Figure 7.56 The total current I across S is the flux of the current density $\mathbf{J}$ across S.

Before continuing our calculations, we comment further regarding the current density field $\mathbf{J}$. The vector field $\mathbf{J}$ at a point is such that its magnitude is the current per unit area at that point, and its direction is that of the current flow. It is not hard to see then that the total current I across an oriented surface S is given by the flux of $\mathbf{J}$; that is,

$$I = \iint_S \mathbf{J}\cdot d\mathbf{S}. \qquad (16)$$

(See Figure 7.56.)

Returning to the magnetic field $\mathbf{B}$ in equation (15), we show that it can be identified as the curl of another vector field $\mathbf{A}$ (to be determined). First, by direct calculation,

$$\nabla_{\mathbf{r}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) = -\frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}.$$

Therefore, equation (15) becomes

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\iiint_D \mathbf{J}(\mathbf{x}) \times \nabla_{\mathbf{r}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) dV. \qquad (17)$$

We rewrite equation (17) using the following standard (and readily verified) identity, where f is a scalar field and $\mathbf{F}$ a vector field (both of class C^2):

$$\nabla \times (f\mathbf{F}) = (\nabla \times \mathbf{F})f - \mathbf{F} \times \nabla f. \qquad (18)$$

Formula (18) is equivalent to

$$\mathbf{F} \times \nabla f = (\nabla \times \mathbf{F})f - \nabla \times (f\mathbf{F}).$$

Therefore,

$$\begin{aligned}\mathbf{J}(\mathbf{x}) \times \nabla_{\mathbf{r}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) &= (\nabla_{\mathbf{r}} \times \mathbf{J}(\mathbf{x}))\frac{1}{\|\mathbf{r}-\mathbf{x}\|} - \nabla_{\mathbf{r}} \times \frac{\mathbf{J}(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|} \\ &= -\nabla_{\mathbf{r}} \times \frac{\mathbf{J}(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|},\end{aligned}$$

since $\mathbf{J}(\mathbf{x})$ is independent of $\mathbf{r}$. Hence,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_D \nabla_{\mathbf{r}} \times \frac{\mathbf{J}(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\,dV = \frac{\mu_0}{4\pi}\nabla_{\mathbf{r}} \times \iiint_D \frac{\mathbf{J}(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\,dV,$$

as $\mathbf{r}$ does not contain any of the variables of integration. Consequently,

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}),$$

where

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_D \frac{\mathbf{J}(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|}\, dV. \tag{19}$$

Thus, $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A})$ and so, by Theorem 4.4 in Chapter 3, we conclude that

$$\nabla \cdot \mathbf{B} = 0. \tag{20}$$

The intuitive content of equation (20) is often expressed by saying that "magnetic monopoles" do not exist.

The vector field $\mathbf{A}$ in equation (19) furnishes an example of a **vector potential** of the field $\mathbf{B}$. (See Exercises 33–38 in the Miscellaneous Exercises for more about vector potentials.)

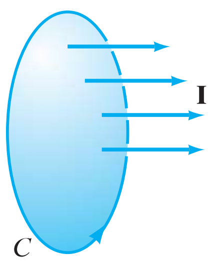

Figure 7.57 The closed loop C is oriented so that it has a right-hand relationship with the direction of current flow it encloses.

Ampère's circuital law If C is a closed loop enclosing a current I, then Ampère's law says that, up to a constant, the current through the loop is equal to the **circulation** of the magnetic field around C. To be precise,

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I. \tag{21}$$

In equation (21), we assume that C is oriented so that C and I are related by a right-hand rule, that is, that they are related in the same way that the orientation of C and the normal to any surface S that C bounds are related in Stokes's theorem. (See Figure 7.57.)

From equation (16) for the total current, equation (21) may be rewritten as

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{S},$$

where S is any (piecewise smooth, oriented) surface bounded by C. Applying Stokes's theorem to the line integral, we obtain

$$\iint_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{S}.$$

Since the loop C and surface S are arbitrary, we conclude that

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{22}$$

Equation (22) is the **differential form of Ampère's law in the static case** (i.e., in the case where $\mathbf{B}$ and $\mathbf{E}$ are constant in time). In the event that the magnetic and electric fields are time varying, we need to make some modifications. The so-called **equation of continuity**, established in Exercise 5 in this section, states that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{23}$$

The difficulty is that if $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ as in equation (22), then equation (23) implies that

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\mu_0 \mathbf{J}) = -\mu_0 \frac{\partial \rho}{\partial t}.$$

However, assuming $\mathbf{B}$ is of class C^2, we must have $\nabla \cdot (\nabla \times \mathbf{B}) = 0$, even in the case where ρ is not constant in time.

The simplest solution to this difficulty is to modify equation (22) by adding an extra term. From Gauss's law, equation (12), we must have

$$\frac{\partial \rho}{\partial t} = \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t}.$$

Thus, if we replace $\mathbf{J}$ by $\mathbf{J} + \epsilon_0(\partial \mathbf{E}/\partial t)$ in equation (22), then we can verify that $\nabla \cdot (\nabla \times \mathbf{B}) = 0$. (See Exercise 16 in this section.) Hence, Ampère's law can be generalized as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{24}$$

The term $\epsilon_0(\partial \mathbf{E}/\partial t)$ in equation (24), known as the **displacement current density**, was first postulated by James Clerk Maxwell in order to generalize Ampère's law to the nonstatic case. (In this context, the original current density field $\mathbf{J}$ is known as the **conduction current density**.)

Equation (24) is not the only possible generalization of equation (22), but it is the simplest one and is consistent with observation. See Exercise 17 in this section for other ways to generalize equation (22) to the nonstatic case.

Faraday's law of induction Michael Faraday observed empirically that the change in magnetic flux across a surface S equals the electromotive force around the boundary C of the surface. This relation can be written as

$$\frac{d\Phi}{dt} = -\oint_C \mathbf{E} \cdot d\mathbf{s}, \tag{25}$$

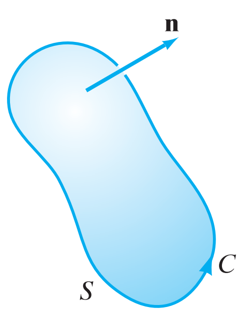

Figure 7.58 The rate of change of magnetic flux across S determines the electromotive force around the boundary C.

where $\Phi(t) = \iint_S \mathbf{B} \cdot d\mathbf{S}$, and C and S are oriented consistently. (See Figure 7.58.) If we apply Stokes's theorem to the line integral, we find that

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S}.$$

Since

$$\frac{d\Phi}{dt} = \frac{d}{dt} \iint_S \mathbf{B} \cdot d\mathbf{S} = \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S},$$

we have

$$-\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S}. \tag{26}$$

Because equation (26) holds for arbitrary surfaces, we conclude that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{27}$$

Summary Equations (12), (20), (24), and (27) together are known as **Maxwell's equations**:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{(Gauss's law);}$$
$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(No magnetic monopoles);}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law);}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(Ampère's law).}$$

Maxwell's equations allow us to reconstruct the electric and magnetic fields from the charge and current densities. They are fundamental to the subject of electricity and magnetism and provide a fitting tribute to the power of the theorems of Stokes and Gauss.

Addendum: Proof of Theorem 4.2

The most obvious idea is to use Green's second formula with

$$g(\mathbf{x}) = \frac{1}{\|\mathbf{r} - \mathbf{x}\|}.$$

However, this function fails to be continuous when $\mathbf{x} = \mathbf{r}$, so Gauss's theorem (and hence Green's formula) cannot be applied so readily. Instead, we need to examine the integrals more carefully.

Throughout the discussion that follows, let S_b denote the sphere of radius b centered at $\mathbf{r}$. First, we establish some subsidiary results.

Lemma 1 If h is a continuous function, then

$$\lim_{b\to 0} \iint_{S_b} \frac{h(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|}\, dS = 0.$$

PROOF The average value of h on S_b is defined to be

$$[h]_{\text{avg}} = \frac{\iint_{S_b} h(\mathbf{x})\, dS}{\text{surface area of } S_b} = \frac{1}{4\pi b^2} \iint_{S_b} h(\mathbf{x})\, dS.$$

(See Exercise 9 of the Miscellaneous Exercises.) Thus,

$$\iint_{S_b} h(\mathbf{x})\, dS = 4\pi b^2 [h]_{\text{avg}}.$$

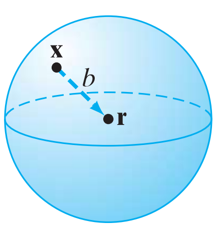

Figure 7.59 If $\mathbf{x}$ is any point on the surface of the sphere of radius b centered at $\mathbf{r}$, then $\|\mathbf{r} - \mathbf{x}\| = b$.

As $\mathbf{x}$ varies over the surface S_b, we have $\|\mathbf{r} - \mathbf{x}\| = b$. (See Figure 7.59.) Hence,

$$\lim_{b\to 0} \iint_{S_b} \frac{h(\mathbf{x})}{\|\mathbf{r} - \mathbf{x}\|}\, dS = \lim_{b\to 0} \iint_{S_b} \frac{1}{b} h(\mathbf{x})\, dS = \lim_{b\to 0} 4\pi b\, [h]_{\text{avg}} = 0. \quad ■$$

To clarify the variables with respect to which we differentiate, let $\nabla_{\mathbf{x}}$ denote the del operator with respect to x, y, and z, and $\nabla_{\mathbf{r}}$ the del operator with respect to $\mathbf{r} = (r_1, r_2, r_3)$.

Lemma 2 With h and S_b as in Lemma 1,

$$\lim_{b\to 0} \oiint_{S_b} h(\mathbf{x})\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right)\cdot d\mathbf{S} = -4\pi h(\mathbf{r}).$$

PROOF Let $\mathbf{n} = (\mathbf{x}-\mathbf{r})/\|\mathbf{r}-\mathbf{x}\|$, the normalization of $\mathbf{x}-\mathbf{r}$. Straightforward calculations yield

$$\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) = -\frac{\mathbf{x}-\mathbf{r}}{\|\mathbf{r}-\mathbf{x}\|^3},$$

and

$$\mathbf{n}\cdot\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) = -\frac{(\mathbf{x}-\mathbf{r})\cdot(\mathbf{x}-\mathbf{r})}{\|\mathbf{r}-\mathbf{x}\|^4} = -\frac{\|\mathbf{x}-\mathbf{r}\|^2}{\|\mathbf{r}-\mathbf{x}\|^4} = -\frac{1}{\|\mathbf{r}-\mathbf{x}\|^2} = -\frac{1}{b^2},$$

for $\mathbf{x}$ on S_b. Then

$$\begin{aligned}\lim_{b\to 0}\oiint_{S_b} h(\mathbf{x})\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) d\mathbf{S} &= \lim_{b\to 0}\oiint_{S_b}\left(h(\mathbf{x})\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right)\cdot\mathbf{n}\right) dS\\ &= \lim_{b\to 0} -\oiint_{S_b}\frac{1}{b^2}h(\mathbf{x})\,dS\\ &= \lim_{b\to 0} -\frac{1}{b^2}\left(4\pi b^2\,[h]_{\mathrm{avg}}\right)\\ &= -4\pi h(\mathbf{r}).\end{aligned}$$

(See the proof of Lemma 1.) ■

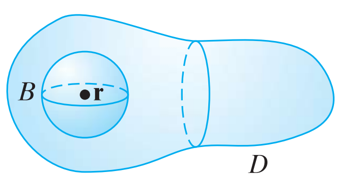

Figure 7.60 The region $D - B$ denotes the solid D with a small ball centered at $\mathbf{r}$ removed.

Returning to the proof of Green's third formula, we look at a region to which we *can* apply Green's second formula, namely, the region $D - B$, where B is a small ball of radius b centered at $\mathbf{r}$. (See Figure 7.60.) By Green's second formula (since $1/\|\mathbf{r}-\mathbf{x}\|$ is not singular on $D - B$), we have

$$\begin{aligned}&\iiint_{D-B}\left(f(\mathbf{x})\nabla^2_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) - \frac{\nabla^2_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\right) dV\\ &\qquad = \iint_{S-S_b}\left(f(\mathbf{x})\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) - \frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\right)\cdot d\mathbf{S}.\end{aligned}\tag{28}$$

By direct calculation $\nabla^2_{\mathbf{x}}(1/\|\mathbf{r}-\mathbf{x}\|) = 0$, so equation (28) becomes

$$\begin{aligned}&\iiint_{D-B} -\frac{\nabla^2_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\,dV\\ &\qquad = \oiint_{S-S_b}\left(f(\mathbf{x})\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) - \frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\right)\cdot d\mathbf{S}.\end{aligned}\tag{29}$$

We may evaluate the right-hand side of equation (29) by replacing the surface integral over $S - S_b$ by separate integrals over S and S_b. Now, we take limits as $b \to 0$. By Lemma 1 with $h(\mathbf{x}) = \nabla_{\mathbf{x}} f(\mathbf{x})\cdot\mathbf{n}$,

$$\oiint_{S_b}\frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\cdot d\mathbf{S} = \oiint_{S_b}\frac{\nabla_{\mathbf{x}} f(\mathbf{x})\cdot\mathbf{n}}{\|\mathbf{r}-\mathbf{x}\|}\,dS \to 0.$$

By Lemma 2,

$$\oiint_{S_b} f(\mathbf{x})\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right)\cdot d\mathbf{S} \to -4\pi f(\mathbf{r}).$$

Since B shrinks to a point as $b \to 0$, we see that equation (29) becomes

$$\iiint_D -\frac{\nabla_{\mathbf{x}}^2 f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\, dV = \oint\!\!\!\oint_S \left(f(\mathbf{x})\nabla_{\mathbf{x}}\left(\frac{1}{\|\mathbf{r}-\mathbf{x}\|}\right) - \frac{\nabla_{\mathbf{x}} f(\mathbf{x})}{\|\mathbf{r}-\mathbf{x}\|}\right)\cdot d\mathbf{S} + 4\pi f(\mathbf{r}),$$

from which Green's third formula follows immediately. ■

7.4 Exercises

1. Prove Green's first formula, stated in Theorem 4.1.

A function $g(x, y, z)$ is said to be ***harmonic*** *at a point (x_0, y_0, z_0) if g is of class C^2 and satisfies Laplace's equation*

$$\nabla^2 g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = 0$$

on some neighborhood of (x_0, y_0, z_0). We say that g is harmonic on a closed region $D \subseteq \mathbf{R}^3$ if it is harmonic at all interior points of D (i.e., not necessarily on the boundary of D). Exercises 2–4 concern some elementary vector analysis of harmonic functions.

2. Assume that D is closed and bounded and that ∂D is a piecewise smooth surface oriented by outward unit normal field $\mathbf{n}$. Let $\partial g/\partial n$ denote $\nabla g \cdot \mathbf{n}$. (The term $\partial g/\partial n$ is called the **normal derivative** of g.) Use Green's first formula with $f(x, y, z) \equiv 1$ to show that, if g is harmonic on D, then

$$\oint\!\!\!\oint_{\partial D} \frac{\partial g}{\partial n}\, dS = 0.$$

3. Let f be harmonic on a region D that satisfies the assumptions of Exercise 2.

(a) Show that

$$\iiint_D \nabla f \cdot \nabla f\, dV = \oint\!\!\!\oint_{\partial D} f \frac{\partial f}{\partial n}\, dS.$$

(b) Suppose $f(x, y, z) = 0$ for all $(x, y, z) \in \partial D$. Use part (a) to show that then we must have $f(x, y, z) = 0$ throughout all of D. (Hint: Think about the sign of $\nabla f \cdot \nabla f$.)

4. Let D be a region that satisfies the assumptions of Exercise 2. Use the result of Exercise 3(b) to show that if f_1 and f_2 are harmonic on D and $f_1(x, y, z) = f_2(x, y, z)$ on ∂D, then, in fact, $f_1 = f_2$ on *all* of D. Thus, we see that harmonic functions are determined by their boundary values on a region. (Hint: Consider $f_1 - f_2$.)

5. (a) Suppose a fluid of density $\rho(x, y, z, t)$ flows with velocity field $\mathbf{F}(x, y, z, t)$ in a solid region W in space enclosed by a smooth surface S. Use Gauss's theorem to show that, if there are no sources or sinks,

$$\nabla \cdot (\rho \mathbf{F}) = -\frac{\partial \rho}{\partial t}.$$

This equation is called the **continuity equation** in fluid dynamics. (Hint: The triple integral $\iiint_W \frac{\partial \rho}{\partial t}\, dV$ is the rate of fluid flowing into W, and the flux of $\rho\mathbf{F}$ across S gives the rate of fluid flowing out of W.)

(b) Use the argument in part (a) to establish the equation of continuity for current densities given in equation (23):

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

Let $T(x, y, z, t)$ denote the temperature at the point (x, y, z) of a solid object D at time t. We define the ***heat flux density*** $\mathbf{H}$ *by $\mathbf{H} = -k\nabla T$. (The constant k is the* ***thermal conductivity****. Note that the symbol ∇ denotes differentiation with respect to x, y, z, not with respect to t.) The vector field $\mathbf{H}$ represents the velocity of heat flow in D. It is a fact from physics that the total heat contained in a solid body D having density ρ and specific heat σ is*

$$\iiint_D \sigma \rho T\, dV.$$

Hence, the total amount of heat leaving D per unit time is

$$-\iiint_D \sigma \rho \frac{\partial T}{\partial t}\, dV.$$

(Here we assume that σ and ρ do not depend on t.) We also know that the heat flux may be calculated as

$$\iint_{\partial D} \mathbf{H} \cdot d\mathbf{S}.$$

Exercises 6–10 concern these notions of temperature, heat, and heat flux density.

6. Use Gauss's theorem to derive the **heat equation**,

$$\sigma \rho \frac{\partial T}{\partial t} = k \nabla^2 T.$$

7. In Exercise 6, suppose that k varies with the points in D; that is, $k = k(x, y, z)$. Show that then we have

$$\sigma \rho \frac{\partial T}{\partial t} = k \nabla^2 T + \nabla k \cdot \nabla T.$$

8. In the heat equation of Exercise 6, suppose that σ, ρ, and k are all constant and the temperature T of the solid D does not vary with time. Show that then T must be harmonic, that is, that $\nabla^2 T = 0$ at all points in the interior of D.

9. (a) If σ, ρ, and k are constant and the temperature T of the solid D is independent of time, show that the (net) heat flux of $\mathbf{H}$ across the boundary of D must be zero.

(b) Let D be the solid region between two concentric spheres of radii 1 and 2. Suppose that the inner sphere is heated to 120° C and the outer sphere to 20° C. Use the result of part (a) to describe the rate of heat flow across the spheres.

10. Consider the three-dimensional heat equation

$$\nabla^2 u = \frac{\partial u}{\partial t} \tag{30}$$

for functions $u(x, y, z, t)$. (Here $\nabla^2 u$ denotes the Laplacian $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2$.) In this exercise, show that any solution $T(x, y, z, t)$ to the heat equation is unique in the following sense: Let D be a bounded solid region in $\mathbf{R}^3$ and suppose that the functions $\alpha(x, y, z)$ and $\phi(x, y, z, t)$ are given. Then there exists a unique solution $T(x, y, z, t)$ to equation (30) that satisfies the conditions

$$T(x, y, z, 0) = \alpha(x, y, z) \quad \text{for } (x, y, z) \in D,$$

and (31)

$$T(x, y, z, t) = \phi(x, y, z, t) \quad \text{for } (x, y, z) \in \partial D \text{ and } t \geq 0.$$

To establish uniqueness, let T_1 and T_2 be two solutions to equation (30) satisfying the conditions in (31) and set $w = T_1 - T_2$.

(a) Show that w must also satisfy equation (30), plus the conditions that

$$w(x, y, z, 0) = 0 \quad \text{for all } (x, y, z) \in D,$$

and

$$w(x, y, z, t) = 0 \quad \text{for all } (x, y, z) \in \partial D \text{ and } t \geq 0.$$

(b) For $t \geq 0$, define the "energy function"

$$E(t) = \frac{1}{2}\iiint_D [w(x, y, z, t)]^2\, dV.$$

Use Green's first formula in Theorem 4.1 to show that $E'(t) \leq 0$ (i.e., that E does not increase with time).

(c) Show that $E(t) = 0$ for all $t \geq 0$. (Hint: Show that $E(0) = 0$ and use part (b).)

(d) Show that $w(x, y, z, t) = 0$ for all $t \geq 0$ and $(x, y, z) \in D$, and thereby conclude the uniqueness of solutions to equation (30) that satisfy the conditions in (31).

11. Show that Ampère's law and Gauss's law imply the continuity equation for $\mathbf{J}$. (Note: In the text, we use the continuity equation to derive Ampère's law.)

12. Suppose that $\mathbf{E}$ is an electric field, in particular, a vector field that satisfies the equation $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. A region D in space is said to be **charge-free** if ρ is zero at all points of D. Describe the charge-free regions of $\mathbf{E} = (x^3 - x)\,\mathbf{i} + \frac{1}{4}y^3\,\mathbf{j} + \left(\frac{1}{9}z^3 - 2z\right)\mathbf{k}$.

13. By considering the derivation of Gauss's law for electric fields, show that, for any inverse square vector field $\mathbf{F}(\mathbf{x}) = k\mathbf{x}/\|\mathbf{x}\|^3$, the flux of $\mathbf{F}$ across a piecewise smooth, closed, oriented surface S is

$$\oiint_S \mathbf{F} \cdot d\mathbf{S} = \begin{cases} 0 & \text{if } S \text{ does not enclose the origin,} \\ 4\pi k & \text{if } S \text{ encloses the origin.} \end{cases}$$

14. Let a point charge of q coulombs be placed at the origin. Recover the formula

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{x}}{\|\mathbf{x}\|^3}$$

by using Gauss's law in the following way:

(a) First, explain that in spherical coordinates, $\mathbf{E}(\mathbf{x}) = E(\mathbf{x})\mathbf{e}_\rho$, that is, that $\mathbf{E}$ has no components in either the $\mathbf{e}_\varphi$- or $\mathbf{e}_\theta$-direction. Next, note that $E(\mathbf{x})$ may be written as $E(\rho)$—that is, that $\|\mathbf{E}\|$ has the same magnitude at all points on a sphere centered at the origin.

(b) Show, using Gauss's law and Gauss's theorem, that

$$\oiint_S E(\rho)\mathbf{e}_\rho \cdot d\mathbf{S} = \frac{q}{\epsilon_0},$$

where S is any smooth, closed surface enclosing the origin.

(c) Now let S be the sphere of radius a centered at the origin. Then the outward unit normal $\mathbf{n}$ to S at (ρ, φ, θ) is $\mathbf{e}_\rho$. Show that

$$\oiint_S E(\rho)\, dS = \frac{q}{\epsilon_0}.$$

(d) Use part (c) to show that $E(\rho) = q/(4\pi\epsilon_0\rho^2)$. Conclude the result desired.

15. (a) Establish the following identity for vector fields $\mathbf{F}$ of class C^2:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}.$$

(Note: $\nabla^2\mathbf{F} = (\nabla \cdot \nabla)\mathbf{F}$.)

(b) In free space (i.e., in the absence of all charges and currents), use Maxwell's equations to show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the wave equation

$$\nabla^2\mathbf{F} = k\frac{\partial^2\mathbf{F}}{\partial t^2},$$

where k is a constant. What is k in each case?

(c) Use part (a), Faraday's law, and Ampère's law to show that

$$\nabla(\nabla \cdot \mathbf{E}) - (\nabla \cdot \nabla)\mathbf{E} = -\mu_0\frac{\partial}{\partial t}\left[\mathbf{J} + \epsilon_0\frac{\partial \mathbf{E}}{\partial t}\right].$$

(d) Show that, in the absence of any charges (i.e., if $\rho = 0$),

$$\nabla^2 \mathbf{E} = \mu_0 \frac{\partial \mathbf{J}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

16. Verify that if the nonstatic version of Ampère's law (equation (24)) holds, then $\nabla \cdot (\nabla \times \mathbf{B}) = 0$.

17. When Maxwell postulated the existence of displacement currents to arrive at a nonstatic version of Ampère's law, he was simply choosing the simplest way to correct equation (22) so that it would be consistent with the continuity equation (23). However, other possibilities are also consistent with the continuity equation.

(a) Show that in order to have equation (22) valid in the static case, then, in general, we must have

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{F}_1}{\partial t}$$

for some (time-varying) vector field $\mathbf{F}_1$ of class C^2.

(b) By taking the divergence of both sides of the equation in part (a), show that

$$\nabla \cdot \frac{\partial \mathbf{F}_1}{\partial t} = \mu_0 \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t}.$$

(c) Use part (b) to argue that, from an entirely *mathematical* perspective, Ampère's law can also be generalized as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{F}_2,$$

where $\mathbf{F}_2$ is any divergence-free vector field. Since no one has observed any physical evidence for $\mathbf{F}_2$'s being nonzero, it is assumed to be zero, as in equation (24).

18. Suppose that $\mathbf{J} = \sigma \mathbf{E}$. (This is a version of Ohm's law that obtains in some electric conductors—here σ is a positive constant known as the **conductivity**.) If $\rho = 0$, show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the so-called **telegrapher's equation**,

$$\nabla^2 \mathbf{F} = \mu_0 \sigma \frac{\partial \mathbf{F}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{F}}{\partial t^2}.$$

19. Let $\mathbf{E}$ and $\mathbf{B}$ be steady-state electric and magnetic fields (i.e., $\mathbf{E}$ and $\mathbf{B}$ are constant in time). The **Poynting vector field** $\mathbf{P} = \mathbf{E} \times \mathbf{B}$ represents radiation flux density. Use Maxwell's equations to show that, for a smooth, orientable, closed surface S bounding a solid region D,

$$\oint\!\!\!\oint_S \mathbf{P} \cdot d\mathbf{S} = -\mu_0 \iiint_D \mathbf{E} \cdot \mathbf{J}\, dV.$$

20. Consider the electric field $\mathbf{E}(\mathbf{r})$ defined by equation (10). Note that the integrals in equation (10) are improper in the sense that they become infinite at points $\mathbf{r} \in D$, where $\rho(\mathbf{r})$ is nonzero. In this exercise, you will show that, nonetheless, the integrals in equation (10) converge when D is a bounded region in $\mathbf{R}^3$ and ρ is a continuous charge density function on D.

(a) Write $\mathbf{E}(\mathbf{r})$ in terms of triple integrals for the individual components. Let $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{x} = (x, y, z)$.

(b) Show that if each component of $\mathbf{E}$ is written in the form $\iiint_D f(\mathbf{x})\, dV$, then $|f(\mathbf{x})| \leq K/\|\mathbf{r} - \mathbf{x}\|^2$, where K is a positive constant.

(c) It follows from part (b) that if

$$\iiint_D \frac{K}{\|\mathbf{r} - \mathbf{x}\|^2}\, dV$$

converges, so must $\iiint_D f(\mathbf{x})\, dV$. Show that

$$\iiint_D \frac{K}{\|\mathbf{r} - \mathbf{x}\|^2}\, dV$$

converges by considering an iterated integral in spherical coordinates with origin at $\mathbf{r}$. (Hint: Look carefully at the integrand in spherical coordinates.)

21. Consider the magnetic field $\mathbf{B}(\mathbf{r})$ defined by equation (15). As was the case with the electric field in equation (10), it is not obvious that the integrals in (15) converge at all $\mathbf{r} \in D$. Follow the ideas of Exercise 20 to show that $\mathbf{B}(\mathbf{r})$ is, in fact, well-defined at all $\mathbf{r}$, assuming a continuous current density field $\mathbf{J}$ and bounded region D in $\mathbf{R}^3$.

True/False Exercises for Chapter 7

1. The function $\mathbf{X}\colon \mathbf{R}^2 \to \mathbf{R}^3$ given by $\mathbf{X}(s, t) = (2s + 3t + 1, 4s - t, s + 2t - 7)$ parametrizes the plane $9x - y - 14z = 107$.

2. The function $\mathbf{X}\colon \mathbf{R}^2 \to \mathbf{R}^3$ given by $\mathbf{X}(s, t) = (s^2 + 3t - 1, s^2 + 3, -2s^2 + t)$ parametrizes the plane $x - 7y - 3z + 22 = 0$.

3. The function $\mathbf{X}\colon (-\infty, \infty) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbf{R}^3$ given by $\mathbf{X}(s, t) = (s^3 + 3\tan t - 1, s^3 + 3, -2s^3 + \tan t)$ parametrizes the plane $x - 7y - 3z + 22 = 0$.

4. The surface $\mathbf{X}(s, t) = (s^2 t, st^2, st)$ is smooth.