Solutions Midterm Exam 2 — Apr. 14, 2017

1. (a) (15 points) Find a matrix S that shows that

$$A = \begin{bmatrix} -49 & 80 \\ -30 & 49 \end{bmatrix} \text{ is similiar to } D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We are seeking an invertible 2×2 matrices, S, so that $AS = SD$. To that end, let

$$S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The equation $AS = SD$ yields the system

$$\begin{cases} -49a + 80c = -a \\ -30a + 49c = -c \\ -49b + 80d = b \\ -30b + 49d = d \end{cases}$$

Notice, this decouples into two systems with two equations and two unknowns

$$\begin{bmatrix} -48 & 80 \\ -30 & 50 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -50 & 80 \\ -30 & 48 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

As

$$\operatorname{rref} \begin{bmatrix} -48 & 80 \\ -30 & 50 \end{bmatrix} = \begin{bmatrix} 1 & -5/3 \\ 0 & 0 \end{bmatrix} \text{ and } \operatorname{rref} \begin{bmatrix} -50 & 80 \\ -30 & 48 \end{bmatrix} = \begin{bmatrix} 1 & -8/5 \\ 0 & 0 \end{bmatrix},$$

a non-trivial solution is

$$S = \begin{bmatrix} 5 & 8 \\ 3 & 5 \end{bmatrix}.$$

Finally, one computes that $\operatorname{rref} S = I_2$ so this matrix is invertible as required.

(b) (5 points) Compute A^{10}.

As $A = SDS^{-1}$, from the above we see that $A^{10} = SD^{10}S^{-1} = SI_2S^{-1} = SS^{-1} = I_2$. Here we used that D was diagonal so

$$D^{10} = \begin{bmatrix} (-1)^{10} & 0 \\ 0 & 1^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

2. Determine which of the following transformations with domain P_2, the space of all polynomials of degree at most 2, is a linear isomorphism. Remember to justify your answers.

(a) (5 points) $T_1 : P_2 \to \mathbb{R}^4$ defined by $T_1(p) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{bmatrix}$.

This map is not a linear isomorphism as $\dim P_2 = 3$ (as a basis is $\{1, x, x^2\}$) while $\dim \mathbb{R}^4 = 4$ and a necessary condition for a linear isomorphism to exist is that the domain and target have the same dimension.

(b) (5 points) $T_2 : P_2 \to \mathbb{R}^3$ defined by $T_3(p) = \begin{bmatrix} p(0) \\ p'(0) \\ p''(0) \end{bmatrix}$.

This map is a linear isomorphism. First of all, it is a linear transformation as

$$T_2(p + kq) = \begin{bmatrix} (p+kq)(0) \\ (p+kq)'(0) \\ (p+kq)''(0) \end{bmatrix} = \begin{bmatrix} p(0) + kq(0) \\ p'(0) + kq'(0) \\ p''(0) + kq''(0) \end{bmatrix} = T_2(p) + kT_2(q)$$

for any $p, q \in P_2, k \in \mathbb{R}$. Second of all,

$$T_2(a_0 + a_1 x + a_2 x^2) = \begin{bmatrix} a_0 \\ a_1 \\ 2a_2 \end{bmatrix}$$

and so an inverse map is given by

$$R_2\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right) = a_0 + a_1 x + \frac{1}{2} a_2 x^2.$$

(c) (5 points) $T_3 : P_2 \to \mathbb{R}^3$ defined by $T_2(p) = \begin{bmatrix} p(1)+2 \\ (p(0))^3 \\ p'(0) \end{bmatrix}$.

This map is not a linear isomorphism as it is not a linear transformation. Indeed, $T_3(0) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \neq \vec{0}$ so the map cannot be linear.

(d) (5 points) $T_4 : P_2 \to P_2$ defined by $T_4(p)(x) = xp'(x)$.

This map is not a linear isomorphism as it is not invertible. Indeed, $T_4(0) = T_4(1) = 0$ so there is not a unique solution to $T_4(x) = 0$.

3. (a) (10 points) Let $\mathbb{R}^{2\times 2}$ be the space of 2×2 matrices and consider the ordered basis, $\mathcal{B}$, of $\mathbb{R}^{2\times 2}$,

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right).$$

For the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ defined by

$$T(A) = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} A - A \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix},$$

determine $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix of T.

Let us write

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We compute

$$T(e_{11}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = e_{12} + e_{21}$$

$$T(e_{12}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = -e_{11} - e_{12} + e_{22}.$$

$$T(e_{21}) = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} = -e_{11} + e_{21} + e_{22}.$$

$$T(e_{22}) = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -e_{12} - e_{21}.$$

Hence,

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

(b) (10 points) Find a basis of $\operatorname{im}(T) \subset \mathbb{R}^{2\times 2}$.

We compute that

$$\operatorname{rref}[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, the first and second columns are the only pivot columns (that is, columns that contain a pivot). Hence, by the algorithm for finding a basis of the image we have that a basis of $\operatorname{im}([T]_{\mathcal{B}}) \subset \mathbb{R}^4$ consists of

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

That is, a basis of $\operatorname{Im}(T) \subset \mathbb{R}^{2\times 2}$ consists of

$$e_{12} + e_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } -e_{11} - e_{12} + e_{22} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}.$$

4. (a) (10 points) Determine *all* $a_1, a_2, a_3 \in \mathbb{R}$ so that the following is an orthogonal matrix:

$$Q = \frac{1}{7}\begin{bmatrix} a_1 & 2 & 6 \\ a_2 & -6 & 3 \\ a_3 & 3 & 2 \end{bmatrix}.$$

Recall, Q is orthogonal if and only if its columns form an orthonormal basis of $\mathbb{R}^3$. Hence, as a first step we must find

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

that are orthogonal to

$$\begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

This means that the a_1, a_2, a_3 satisfy the system

$$\begin{cases} 2a_1 - 6a_2 + 3a_3 = 0 \\ 6a_1 + 3a_2 + 2a_3 = 0 \end{cases}$$

That is,

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \ker \begin{bmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \end{bmatrix}$$

Computing,

$$\operatorname{rref}\begin{bmatrix} 2 & -6 & 3 \\ 6 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

we see that

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = k\begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}$$

for some $k \in \mathbb{R}$. As the first column has to also have length 1 and

$$\left\|\frac{1}{7}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right\|^2 = \frac{1}{49}k^2((-3)^2 + 2^2 + 6^2) = k^2$$

we see that $k = \pm 1$. Hence, the possible choices for a_1, a_2, a_3, are

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \pm\begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}.$$

Finally, to check that this is truly an orthgonal matrix we need to see that the second and third columns are orthogonal and of unit length. This is straightforward.

(b) (10 points) Suppose a matrix, M, has the QR-factorization $M = QR$. Determine R given

$$M = \begin{bmatrix} 4 & -1 & 0 \\ 4 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} \text{ and } Q = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ 1 & -2 & -2 \end{bmatrix}.$$

We observe that

$$M = QR \Rightarrow R = Q^{-1}M = Q^{\top}M$$

where the second equality follows from one of the properties of orthogonal matrices. Hence,

$$M = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -1 & 2 & -2 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} 4 & -1 & 0 \\ 4 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 18 & -3 & -3 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. In what follows, determine if the matrix C is symmetric, skew-symmetric or if there is not enough information to decide. Remember to justify your answer

(a) (5 points) $C = QAQ^{-1}$ where $A \in \mathbb{R}^{n\times n}$ is symmetric and $Q \in \mathbb{R}^{n\times n}$ is orthogonal.

Recall, that one of the properties of orthogonal matrices is that $Q^\top = Q^{-1}$. We compute using properties of the transpose that

$$C^\top = (QAQ^{-1})^\top = (Q^{-1})^\top A^\top Q^\top$$

Hence, using properties of orthogonal matrices and the fact that A is symmetric we have,

$$C^\top = (Q^\top)^\top AQ^\top = QAQ^\top = QAQ^{-1} = C.$$

As such, C is symmetric.

(b) (5 points) $C = ABA$, where $A, B \in \mathbb{R}^{n\times n}$ are both skew-symmetric.

We compute using properties of the transpose and the fact that both A and B are skew-symmetric that

$$C^\top = (ABA)^\top = A^\top B^\top A^\top = (-A)(-B)(-A) = -ABA = -C.$$

Hence, C is skew-symmetric.

(c) (5 points) $C = A^\top A - AA^\top$, where $A \in \mathbb{R}^{n\times n}$

We compute using properties of the transpose that

$$C^\top = (A^\top A - AA^\top)^\top = (A^\top A)^\top - (AA^\top)^\top = A^\top (A^\top)^\top - (A^\top)^\top A^\top = A^\top A - AA^\top = C.$$

Hence, C is symmetric.

(d) (5 points) $C = I_n + P^2$ where $P \in \mathbb{R}^{n\times n}$.

We compute

$$C^\top = (I_n + P^2)^\top = I_n^\top + (P^2)^\top = I_n + (P^\top)^2.$$

Without further information about P, we cannot decide.