Solutions to Math 201 Midterm II Spring 11

1. (a) T maps

$$\begin{array}{ccc} c_1\cdot 1 + c_2\cdot x + c_3\cdot x^2 & \xrightarrow{T} & d_1\cdot 1 + d_2\cdot x + d_3\cdot x^2 \\ \downarrow & & \downarrow \\ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} & \xrightarrow{\text{multiply by } A} & \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \end{array}$$

Since

$$T(1) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2,$$
$$T(x) = x = 0\cdot 1 + 1 : x + 0\cdot x^2,$$
$$T(x^2) = x^2 + 2 = 2\cdot 1 + 0\cdot x + 1\cdot x^2,$$

we have

$$A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

which gives the three columns of A so $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Next we want to find the expression for T in the new basis

$$\begin{array}{ccc} a_1\cdot(1+x) + a_2\cdot(x+x^2) + a_3\cdot(1+x^2) & \xrightarrow{T} & b_1\cdot(1+x) + b_2\cdot(x+x^2) + b_3\cdot(1+x^2) \\ \downarrow & & \downarrow \\ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} & \xrightarrow{\text{multiply by } B} & \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \end{array}$$

Since we already know the matrix for B in the standard coordinates the easiest way to get B is to change coordinates

$$a_1\cdot(1+x) + a_2\cdot(x+x^2) + a_3\cdot(1+x^2) = c_1\cdot 1 + c_2\cdot x + c_3\cdot x^2$$

Since the new coordinates already are expressed in terms of the old ones, the easiest way is to get the matrix S from the new coordinates to the old ones $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = S\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$. We get

$$S\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad S\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad S\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{so} \quad S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Multiplication B corresponds to $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \xrightarrow{\text{multiply by } S} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \xrightarrow{\text{multiply by } A} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \xrightarrow{\text{multiply by } S^{-1}} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

so $B = S^{-1}AS = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = ...$ For more info see Ex 5, Ex 8 in sec 4.3.

2. (a) FALSE. Similar matrices have the same determinant: $\det \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3$ whereas $\det \left(S^{-1} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} S \right) = \det S^{-1} \det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \det S = \det \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = 0$, since $\det S^{-1} \det S = \det S^{-1}S = 1$.

(b) FALSE. T can be a linear transformation of rank 0 or 1.

3. The kernel of the orthogonal projection onto V is the orthogonal complement of V, i.e. $V^{\perp} = \{\mathbf{x};\ \mathbf{x} \cdot \mathbf{v} = 0, \text{ for every } \mathbf{v} \in V\}$. Since $V = \text{Span}\,\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, it follows that $V^{\perp} = \{\mathbf{x};\ \mathbf{x} \cdot \mathbf{v}_1 = 0, \text{ and } \mathbf{x} \cdot \mathbf{v}_2 = 0\} = \text{Ker}\, A$, where $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$.

Solving the system $A\mathbf{x} = \mathbf{0}$ gives $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_4 - x_3 \\ -2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, where x_3, x_4 are free. It follows that $\mathbf{w}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\text{Ker}(\text{Proj}_V)$.

We want to use Gram-Schmidt to construct and orthonormal basis $\mathbf{u}_1, \mathbf{u}_2$ from $\mathbf{w}_1, \mathbf{w}_2$.

First we get an orthonormal vector $\mathbf{u}_1 = \mathbf{w}_1 / \|\mathbf{w}_1\| = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$. Then we calculate the orthogonal projection of $\mathbf{w}_2$ onto the span of $\mathbf{u}_1$ to be $\mathbf{p}_1 = (\mathbf{w}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 = -2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

$\mathbf{w}_2 - \mathbf{p}_1$ is now orthogonal to $\mathbf{u}_1$ so we just have to normalize it $\mathbf{u}_2 = (\mathbf{w}_2 - \mathbf{p}_1)/\|\mathbf{w}_2 - \mathbf{p}_1\| = ...$

4. (a) TRUE. A matrix Q is called orthogonal if $Q^TQ = I$ and the transpose of a product satisfy $(AB)^T = A^TB^T$. Inverses of orthogonal matrices are orthogonal. In fact $S^{-1} = S^T$. Product of orthogonal matrices are orthogonal. In fact, $(S^{-1}AS)^T = S^TA^T(S^{-1})^T$ so $(S^{-1}AS)^TS^{-1}AS = S^TA^T(S^{-1})^TS^{-1}AS = S^TA^TIAS = S^TS = I$.

(b) FALSE. A is scaling by 2, B is scaling by $1/2$. Then BA is the identity transformation.

5. The so called least squares 'solution' to the system $A\mathbf{x} = \mathbf{b}$ is only an approximate solution to this system but instead it is the solution to the normal equation $A^TA\mathbf{x} = A^T\mathbf{b}$, where $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The normal equation is $\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$ so $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8/5 \end{bmatrix}$.

The orthogonal projection of $\mathbf{b}$ onto the subspace $\operatorname{Im} A$ is $A\mathbf{x} = \begin{bmatrix} 1 \\ 8/5 \\ 16/5 \end{bmatrix}$, because the least square 'solution' is the $\mathbf{x}$ that makes $A\mathbf{x}$ as close as possible to $\mathbf{b}$ and that is the orthogonal projection of $\mathbf{b}$ onto the image of A.

For more info see Ex 1 in sec 5.4.