Name: ______________________________ April 11, 2006

Section Number/Time ____________________

TA ____________________

MATH 201, MIDTERM #2

Directions: This is a pencil-and-paper exam. You are asked to put away all books, notes, calculators, cell phones, and other computing and/or telecommunications equipment. The last page of this booklet is blank and is intended for use as scrap paper. Additional sheets of paper are available upon request.

Grading: There are fifteen questions on the exam, typically worth 4 points, for a total of 60 points.

Special Note: Many of the problems in this exam may be interrelated. If the answer to one question appears to require the answer to a previous question which you have not solved, you may instead explain how this missing information would be used to solve the problem.

Problems 1-4 will involve the linear space $V = \left\{ 2 \times 2 \text{ matrices } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$.

1. [6 points] Find a basis for V.

2. [2 points] What is the dimension of V?

3. Given any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can "rotate" the entries to form a new matrix $T(A) = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$.

Find a matrix A which solves $T(A) = -A$.

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We continue examining the linear transformation $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$

4. Suppose we know that $T^4(A) = A$ for every 2×2 matrix A.
Explain why every eigenvalue of T must satisfy $\lambda^4 = 1$.

In Problems 5 and 6, we consider the linear transformation $Tf(x) = xf(x) - 3\int_0^x f(t)\,dt,$

acting on the space $\mathcal{P}_2 = \Big\{\text{All quadratic polynomials } f(x) = ax^2 + bx + c\Big\}.$

5. If g represents the function $g(x) = 1$ for all x, what is the function Tg?

6. Show that if f is any function in $\mathcal{P}_2$, then the resulting function Tf is also in $\mathcal{P}_2$.

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7. What is the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 10 \end{bmatrix}$?

8. Now consider $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 6 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 10 \end{bmatrix}$. How is the determinant of B related to the determinant of A?
Hint: Use row operations to turn matrix B into matrix A.

9. Now consider $C = \begin{bmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & 5 & 7 \\ 0 & 2 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. How is the determinant of C related to the determinant of A?

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10. What are the eigenvalues of the matrix $M = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix}$? Call them λ_1 and λ_2.

Your answer may include complex numbers, if the eigenvalues are complex.

Check your work carefully! You will have trouble with the next few questions if there is a mistake here.

11. Write down a matrix whose kernel tells you the eigenvectors of M associated to the eigenvalue λ_1.

Again, your answer may have some complex numbers in it.

12. Find an eigenvector $\vec{v}_1$ of M with eigenvalue λ_1.

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On the previous page we found that the matrix $M = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix}$ has the eigenvalues $\lambda_1 =$ ___ and $\lambda_2 =$ ___ .

13. How large are the entries of the matrix M^{50}?
First give an approximate answer in terms of the eigenvalues of M.

Then circle the most accurate description from the list below:

a) Larger than 1,000,000
b) Between 1 and 1,000,000
c) Between 0.000001 and 1
d) Smaller than 0.000001
e) It depends which entry of the matrix M^{50} we're looking at.

14. What are the eigenvalues of the matrix $A = 2\begin{bmatrix} \cos 72^\circ & -\sin 72^\circ \\ \sin 72^\circ & \cos 72^\circ \end{bmatrix}$?
Your answer may be expressed in terms of trigonometric functions.

15. Fill in the blanks: $A^5 = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$.

[Here, A is the same matrix as in Problem 14.]