MATH 201, MIDTERM #2

Directions: This is a pencil-and-paper exam. You are asked to put away all books, notes, calculators, cell phones, and other computing and/or telecommunications equipment. The last page of this booklet is blank and is intended for use as scrap paper. Additional sheets of paper are available upon request.

Grading: There are fifteen questions on the exam, typically worth 4 points, for a total of 60 points.

Special Note: Many of the problems in this exam may be interrelated. If the answer to one question appears to require the answer to a previous question which you have not solved, you may instead explain how this missing information would be used to solve the problem.

Problems 1-4 will involve the linear space \( V = \left\{ 2 \times 2 \text{ matrices} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\} \).

1. [6 points] Find a basis for \( V \).

2. [2 points] What is the dimension of \( V \)?

3. Given any \( 2 \times 2 \) matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), we can “rotate” the entries to form a new matrix \( T(A) = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \).

Find a matrix \( A \) which solves \( T(A) = -A \).
We continue examining the linear transformation \( T\left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} c & a \\ d & b \end{bmatrix} \)

4. Suppose we know that \( T^4(A) = A \) for every \( 2 \times 2 \) matrix \( A \).
   Explain why every eigenvalue of \( T \) must satisfy \( \lambda^4 = 1 \).

In Problems 5 and 6, we consider the linear transformation \( Tf(x) = xf(x) - 3 \int_0^x f(t) \, dt \),
acting on the space \( P_2 = \left\{ \text{All quadratic polynomials } f(x) = ax^2 + bx + c \right\} \).

5. If \( g \) represents the function \( g(x) = 1 \) for all \( x \), what is the function \( Tg \)?

6. Show that if \( f \) is any function in \( P_2 \), then the resulting function \( Tf \) is also in \( P_2 \).
7. What is the determinant of the matrix \( A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 10 \end{bmatrix} \)?

8. Now consider \( B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 6 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 10 \end{bmatrix} \). How is the determinant of \( B \) related to the determinant of \( A \)?

*Hint:* Use row operations to turn matrix \( B \) into matrix \( A \).

9. Now consider \( C = \begin{bmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & 5 & 7 \\ 0 & 2 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix} \). How is the determinant of \( C \) related to the determinant of \( A \)?
10. What are the eigenvalues of the matrix \( M = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix} \)? Call them \( \lambda_1 \) and \( \lambda_2 \).

   Your answer may include complex numbers, if the eigenvalues are complex.

   Check your work carefully! You will have trouble with the next few questions if there is a mistake here.

11. Write down a matrix whose kernel tells you the eigenvectors of \( M \) associated to the eigenvalue \( \lambda_1 \).

   Again, your answer may have some complex numbers in it.

12. Find an eigenvector \( \vec{v}_1 \) of \( M \) with eigenvalue \( \lambda_1 \).
On the previous page we found that the matrix \( M = \begin{bmatrix} -1 & -3 \\ 4 & 5 \end{bmatrix} \) has the eigenvalues \( \lambda_1 = \quad \) and \( \lambda_2 = \quad \).

13. How large are the entries of the matrix \( M^{50} \)?
   First give an approximate answer in terms of the eigenvalues of \( M \).

Then circle the most accurate description from the list below:
   a) Larger than 1,000,000
   b) Between 1 and 1,000,000
   c) Between 0.000001 and 1
   d) Smaller than 0.000001
   e) It depends which entry of the matrix \( M^{50} \) we're looking at.

14. What are the eigenvalues of the matrix \( A = 2 \begin{bmatrix} \cos 72^\circ & -\sin 72^\circ \\ \sin 72^\circ & \cos 72^\circ \end{bmatrix} \)?
   Your answer may be expressed in terms of trigonometric functions.

15. Fill in the blanks: \( A^5 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \).
   [Here, \( A \) is the same matrix as in Problem 14.]