Midterm1 – 201, Fall 2016

Instructor: Wenjing Liao

- Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted.
- No notes, books or calculators are allowed.
- Read each problem carefully. Show all work for full credit.
- Make sure you have **10 pages**, including the cover page (Page 1) and the score page (Page 2).

Name: ________________ **Section:** ________________

I would not assist anybody else in the completion of this exam. I would not copy answers from others. I would not have another student take the exam for me.

Signature: ________________

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(3)	
(4)	
(5)	
(6)	
Total	

Problem 1: (20 points):

Write down T if the following statement is true and F if the statement is false. (2 points each)

Example: F 10 vectors in $\mathbb{R}^9$ can be linearly independent.

(1) T The transformation $T(A) = A^T$ from $\mathbb{R}^{5\times 5}$ to $\mathbb{R}^{5\times 5}$ is an isomorphism.

(2) F Consider the linear transformation:

$$T(\vec{x}) = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_{n-1} & \vec{x} \end{bmatrix}$$

from $\mathbb{R}^n$ to $\mathbb{R}$, where $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$ are linearly independent vectors in $\mathbb{R}^n$. The nullity of T is equal to n.

(3) T Determinant of the following matrix is necessarily 0.

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}$$

(4) F The dimension of $W^\perp$ is equal to 2, where

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \end{bmatrix} \right).$$

(5) F The following matrix is an orthogonal matrix.

$$\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

(6) F If the $n \times n$ matrices A and B are orthogonal matrices, $A + B$ must be orthogonal as well.

(7) T Consider an orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in $\mathbb{R}^n$. The least-squares solution of the system $A\vec{x} = \vec{u}_n$ where $A = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}]$ is $\vec{x} = \vec{0}$.

(8) T $\langle f, g \rangle = f(1)g(1) + f(2)g(2)$ is an inner product in P_1.

(9) T Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, then $\lambda^2 + \lambda + 1$ is an eigenvalue of the matrix $A^2 + A + I_n$.

(10) F There is an orthogonal transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

Problem 2 (12 points):

Let
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(1) Find the eigenvalues of A. (5 points)

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$$

Characteristic polynomial $f_A(\lambda) = \det(A - \lambda I) = (1-\lambda)\left[(1-\lambda)^2 - 1\right]$
$= (1-\lambda)(\lambda^2 - 2\lambda) = (1-\lambda)\lambda(\lambda-2)$

Eigenvalues $\lambda_1 = 1$ $\quad \lambda_2 = 0$ $\quad \lambda_3 = 2.$

(2) Find the eigenvectors of A. (7 points)

For $\lambda_1 = 1$ $\quad E_{\lambda_1} = \ker(A - \lambda_1 I) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 + x_3 = 0 \\ x_2 = 0 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \end{bmatrix}$$

For $\lambda_2 = 0$ $\quad E_{\lambda_2} = \ker(A - \lambda_2 I) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 = 0 \\ x_2 + x_3 = 0 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ -x_2 \end{bmatrix}$$

For $\lambda_3 = 2$ $\quad E_{\lambda_3} = \ker(A - \lambda_3 I) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} x_1 = 0 \\ x_2 - x_3 = 0 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix}$$

Problem 3 (16 points):

Let

$$A = \begin{array}{c} \begin{array}{cc} \vec{V}_1 & \vec{V}_2 \end{array} \\ \begin{bmatrix} -1 & 1 \\ 1 & 3 \\ 1 & 3 \\ -1 & 1 \end{bmatrix} \end{array}.$$

(1) Find the QR factorization of A. (10 points)

$r_{11} = \|\vec{V}_1\| = 2 \qquad \vec{u}_1 = \frac{\vec{V}_1}{r_{11}} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$

$r_{12} = \vec{u}_1 \cdot \vec{V}_2 = 2 \qquad \vec{V}_2^{\perp} = \vec{V}_2 - r_{12}\vec{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$

$r_{22} = \|\vec{V}_2^{\perp}\| = 4 \qquad \vec{u}_2 = \frac{\vec{V}_2^{\perp}}{r_{22}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$A = \overbrace{\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}}^{Q} \overbrace{\begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}}^{R}$$

(2) Let $V = \text{image}(A)$. Compute $\text{Proj}_V \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$. (6 points)

Let $\vec{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

$$\begin{aligned} \text{Proj}_V \vec{x} &= (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2 \\ &= \tfrac{1}{2}(-4+3+2-1)\vec{u}_1 + \tfrac{1}{2}(4+3+2+1)\vec{u}_2 \\ &= 5\vec{u}_2 \\ &= \frac{5}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Problem 4 (20 points, 5 points each):

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $B = \begin{pmatrix} a & d & g \\ b & e & h \\ 1 & 2 & 3 \end{pmatrix}$ such that $\det A = 2$ and $\det B = 3$.

(1) Compute $\det \begin{pmatrix} 0 & 0 & 1 & -2 \\ a & b & c & 1 \\ d & e & f & 2 \\ g & h & i & 3 \end{pmatrix}$.

$$= (-1)^{1+3} \det \begin{pmatrix} a & b & 1 \\ d & e & 2 \\ g & h & 3 \end{pmatrix} + (-1)^{1+4}(-2) \det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$= 3 + 2(2) = 7$$

(2) Suppose $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the solution of $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Evaluate x_3.

$$x_3 = \frac{\det \begin{pmatrix} a & b & 1 \\ d & e & 2 \\ g & h & 3 \end{pmatrix}}{\det A} = \frac{\det B}{\det A} = \frac{3}{2}.$$

(3) Let $C = (ABA^{-1})^{2016}$. Compute $\det C$.

$$\det C = \left(\det(ABA^{-1})\right)^{2016} = \left(\det A \det B \frac{1}{\det A}\right)^{2016}.$$

$$= 3^{2016}$$

(4) Compute $\det E$, where E is the reduced row echelon form of B.

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det E = 1$$

Problem 5 (20 points):

Consider the linear transformation $T : P_3 \to P_3$ such that
$T(a + bt + ct^2 + dt^3) = a + dt + (b + c)t^2 + (a + d)t^3$.

(1) Given the basis $\mathfrak{B} = \{1, t, t^2, t^3\}$ in P_3. Find the $\mathfrak{B}$-matrix of T with respect to the basis $\mathfrak{B}$. (8 points)

$f_1(t) = 1 \qquad f_2(t) = t \qquad f_3(t) = t^2 \qquad f_4(t) = t^3$

$$B = \begin{bmatrix} [Tf_1]_B & [Tf_2]_B & [Tf_3]_B & [Tf_4]_B \end{bmatrix}$$

$Tf_1 = 1 + t^3$

$Tf_2 = t^2$

$Tf_3 = t^2$

$Tf_4 = t + t^3$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

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Basis of $\ker T = \{ 0 \cdot f_1 + 1 \cdot f_2 + (-1) f_3 + 0 \cdot f_4 \}$

$= \{ t - t^2 \}$

(2) Find a basis of the kernel of T. (6 points)

$$\ker B = \{ \vec{x} : B\vec{x} = \vec{0} \} = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \begin{matrix} x_1 = 0 & x_4 = 0 \\ x_2 + x_3 = 0 \end{matrix} \right\}.$$

$$= \left\{ \begin{bmatrix} 0 \\ x_2 \\ -x_2 \\ 0 \end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Basis of $\ker B$: $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$

Basis of $\ker T$: $L_B^{-1} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ where L_B is the coordinate map.

(3) Find a basis of the image of T. (6 points)

Basis of Image B : $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

Basis of Image T : $\{ 1 + t^3, t^2, t + t^3 \}$.

Second way to do (2).

$$\ker T = \left\{ a + bt + ct^2 + dt^3 : \begin{matrix} a = 0 & d = 0 \\ b + c = 0 & a + d = 0 \end{matrix} \right\}.$$

$$= \{ bt - bt^2 \}$$

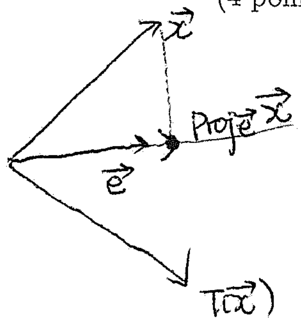
$$= \text{Span} \{ t - t^2 \}$$

Basis of $\ker T$: $t - t^2$.

Problem 6 (12 points):

Let $\vec{e}$ be a unit vector in $\mathbb{R}^3$. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\vec{x}) = 2(\vec{e} \cdot \vec{x})\vec{e} - \vec{x}$.

(1) Given a geometric interpretation of T. In other words, express T in terms of reflections, projections, or rotations. (4 points)



$T\vec{x}$ is the reflection with respect to the line in the direction of $\vec{e}$

(2) Is T an isomorphism? Why? (4 points)

Yes, T is linear $T(\vec{x}_1+\vec{x}_2) = 2[\vec{e}\cdot(\vec{x}_1+\vec{x}_2)]\vec{e} - (\vec{x}_1+\vec{x}_2)$

$$= (2\vec{e}\cdot\vec{x}_1 - \vec{x}_1) + (2\vec{e}\cdot\vec{x}_2 - \vec{x}_2)$$

$$= T\vec{x}_1 + T\vec{x}_2$$

$$T(k\vec{x}) = k[2(\vec{e}\cdot\vec{x})\vec{e} - \vec{x}] = kT\vec{x}$$

T is invertible : $T(T\vec{x}) = \vec{x}$ So the inverse of T is T itself.

(3) Is T an orthogonal transformation? Why? (4 points)

Yes, reflection preserves the length of vectors.