Midterm 1 – 201, Fall 2016
Instructor: Wenjing Liao

- Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted.
- No notes, books or calculators are allowed.
- Read each problem carefully. Show all work for full credit.
- Make sure you have 10 pages, including the cover page (Page 1) and the score page (Page 2).

Name: ________________  Section: _______________

I would not assist anybody else in the completion of this exam. I would not copy answers from others. I would not have another student take the exam for me.

Signature: _______________
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Problem 1: (20 points):

Write down T if the following statement is true and F if the statement is false. (2 points each)

Example: F 10 vectors in \( \mathbb{R}^9 \) can be linearly independent.

(1) ______ The transformation \( T(A) = A^T \) from \( \mathbb{R}^{5 \times 5} \) to \( \mathbb{R}^{5 \times 5} \) is an isomorphism.

(2) ______ Consider the linear transformation:

\[
T(\vec{x}) = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_{n-1} & \vec{x} \end{bmatrix}
\]

from \( \mathbb{R}^n \) to \( \mathbb{R} \), where \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-1} \) are linearly independent vectors in \( \mathbb{R}^n \). The nullity of \( T \) is equal to \( n \).

(3) ______ Determinant of the following matrix is necessarily 0.

\[
\begin{bmatrix}
 a_1 b_1 & a_1 b_2 & a_1 b_3 & a_1 b_4 \\
 a_2 b_1 & a_2 b_2 & a_2 b_3 & a_2 b_4 \\
 a_3 b_1 & a_3 b_2 & a_3 b_3 & a_3 b_4 \\
 a_4 b_1 & a_4 b_2 & a_4 b_3 & a_4 b_4
\end{bmatrix}
\]

(4) ______ The dimension of \( W^\perp \) is equal to 2, where

\[
W = \text{span} \begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 3 & -3 \\ 4 & -4 \end{pmatrix}
\]

(5) ______ The following matrix is an orthogonal matrix.

\[
\begin{bmatrix}
 2 & -2 & 1 \\
 1 & 2 & 2 \\
 2 & 1 & -2
\end{bmatrix}
\]
(6) _____ If the $n \times n$ matrices $A$ and $B$ are orthogonal matrices, $A + B$ must be orthogonal as well.

(7) _____ Consider an orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in $\mathbb{R}^n$. The least-squares solution of the system $A\vec{x} = \vec{u}_n$ where $A = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}]$ is $\vec{x} = \vec{0}$.

(8) _____ $(f, g) = f(1)g(1) + f(2)g(2)$ is an inner product in $P_1$.

(9) _____ Let $A$ be an $n \times n$ matrix. If $\lambda$ is an eigenvalue of $A$, then $\lambda^2 + \lambda + 1$ is an eigenvalue of the matrix $A^2 + A + I_n$.

(10) _____ There is an orthogonal transformation $T$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

\[
T \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.
\]
Problem 2 (12 points):

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(1) Find the eigenvalues of $A$. (5 points)

(2) Find the eigenvectors of $A$. (7 points)
Problem 3 (16 points):

Let

\[ A = \begin{bmatrix} -1 & 1 \\ 1 & 3 \\ -1 & 1 \end{bmatrix} \]

(1) Find the QR factorization of \( A \). (10 points)

(2) Let \( V = \text{image}(A) \). Compute \( \text{Proj}_V \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \). (6 points)
Problem 4 (20 points, 5 points each):

Let \( A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \) and \( B = \begin{pmatrix} a & d & g \\ b & e & h \\ 1 & 2 & 3 \end{pmatrix} \) such that \( \det A = 2 \) and \( \det B = 3 \).

(1) Compute \( \det \begin{pmatrix} 0 & 0 & 1 & -2 \\ a & b & c & 1 \\ d & e & f & 2 \\ g & h & i & 3 \end{pmatrix} \).

(2) Suppose \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \) is the solution of \( A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \). Evaluate \( x_3 \).

(3) Let \( C = (ABA^{-1})^{2019} \). Compute \( \det C \).

(4) Compute \( \det E \), where \( E \) is the reduced row echelon form of \( B \).
Problem 5 (20 points):

Consider the linear transformation \( T : P_3 \to P_3 \) such that
\[
T(a + bt + ct^2 + dt^3) = a + dt + (b + c)t^2 + (a + d)t^3.
\]

(1) Given the basis \( \mathcal{B} = \{1, t, t^2, t^3\} \) in \( P_3 \). Find the \( \mathcal{B} \)-matrix of \( T \) with respect to the basis \( \mathcal{B} \). (8 points)
(2) Find a basis of the kernel of $T$. (6 points)

(3) Find a basis of the image of $T$. (6 points)
Problem 6 (12 points):

Let \( \vec{e} \) be a unit vector in \( \mathbb{R}^3 \). Consider the linear transformation \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) such that \( T(\vec{x}) = 2(\vec{e} \cdot \vec{x})\vec{e} - \vec{x} \).

1. Given a geometric interpretation of \( T \). In other words, express \( T \) in terms of reflections, projections, or rotations. (4 points)

2. Is \( T \) an isomorphism? Why? (4 points)

3. Is \( T \) an orthogonal transformation? Why? (4 points)