THE JOHNS HOPKINS UNIVERSITY
Krieger School of Arts and Sciences
SECOND MIDTERM EXAM - FALL 2005
110.201 – LINEAR ALGEBRA

Instructor: Professor Carel Faber
Duration: 50 minutes   November 22, 2005

No calculators allowed   **Total = 100 points**

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SECTION (weekday and time): 1, 2, 3, 4

ETHICS PLEDGE:

I agree to complete this examination without unauthorized assistance from any person, materials, or device.

SIGNATURE: DATE:

1. [**25** points] Let P_1 be the linear space of polynomials $f(t)$ of degree ≤ 1. Let T from P_1 to P_1 be the linear transformation given by

$$T(-1) = -5 - 2t \qquad \text{and} \qquad T(1+2t) = -3.$$

(a) [**7** points] Find the matrix A of T with respect to the standard basis $\mathcal{A} = (1, t)$.

$T(-1) = -T(1)$ So $T(1) = -T(-1) = 5+2t$

$T(2t) = T(-1) + T(1+2t) = -5-2t-3 = -8-2t$, so

$T(t) = \frac{1}{2} T(2t) = -4-t.$

So 1^{st} col. of $A = [T(1)]_{\mathcal{A}} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$,

2^{nd} col. of $A = [T(t)]_{\mathcal{A}} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$, so $A = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$.

(b) [**8** points] Find the matrix B of T with respect to the basis $\mathcal{B} = (1+t, 2+t)$.

$T(1+t) = T(1) + T(t) = 1+t$ (hey!)

$T(2+t) = T(2) + T(t) = 10+4t-4-t = 6+3t = 3(2+t)$ (hey!).

As above: 1^{st} col. of $B = [T(1+t)]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2^{nd} col. of $B = [T(2+t)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, so $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$.

[Note: A has been diagonalized w.r.t. the basis $\mathcal{B}$, it is an eigenbasis!]

(c) [**5** points] Find the change of basis matrix S from the basis $\mathcal{B}$ to the basis $\mathcal{A}$.

$$\left.\begin{array}{l} 1^{st} \text{ col. of } S = [1+t]_{\mathcal{A}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ 2^{nd} \text{ col. of } S = [2+t]_{\mathcal{A}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{array}\right\} S = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

Note: S is made from eigenvectors for A in this case, since B is diagonal.

Easy to check: $AS = SB$.

(d) [**5** points] Is SBS^{-1} equal to A? Motivate your answer.

① by computation: $S^{-1} = \frac{1}{\det(S)} \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$

$BS^{-1} = \cdots = \begin{pmatrix} -1 & 2 \\ 3 & -3 \end{pmatrix}$

$SBS^{-1} = \cdots = \begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix} = A$, Yes!

② If you understood what happened in (b) and (c), then $AS = SB$ so $SBS^{-1} = A$. (← Preferred solution.)

2. [**25** points] Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 7 & 0 \\ 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 3 & 4 & 5 & 2 & 6 \end{bmatrix}.$$

$$\det A = (+3) \begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 5 & 6 \end{vmatrix}$$ (develop along the 4^{th} row)

$$= 3 \cdot (+2) \cdot \begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{vmatrix}$$ (develop along the 2^{nd} row)

$$= 6\left(1 \cdot 4 \cdot 6 + 2 \cdot 5 \cdot 1 + 3 \cdot 0 - 1 \cdot 4 \cdot 3 - 0 - 0 \right)$$

$$= 6(24 + 10 - 12) = 6 \cdot 22 = 132.$$

3. [**25** points] Consider the linear space P_1 of polynomials of degree ≤ 1 with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

(a) [**5** points] Determine the norm of the element $f(t) = 1$ of P_1.

By def., norm of $1 = \|1\| = \sqrt{\langle 1,1 \rangle}$.

Now $\langle 1,1 \rangle = \int_0^1 1 \cdot 1 \, dt = \int_0^1 1 \, dt = [t]_0^1 = 1 - 0 = 1$.

So $\|1\| = \sqrt{1} = 1$.

(Do not take this for granted! With $\int_0^2 f(t)g(t)dt$,

$\|1\| = \sqrt{2}$!)

(b) [**5** points] Show that $g(t) = 2t - 1$ is orthogonal to $f(t)$.

$\langle g(t), f(t) \rangle = \langle 2t-1, 1 \rangle = \int_0^1 (2t-1)\, dt = [t^2 - t]_0^1$

$= 1 - 1 - 0 + 0 = 0$. QED.

(c) [**5** points] Determine the norm of the element $g(t)$ of P_1.

$$\langle g(t), g(t)\rangle = \int_0^1 (2t-1)^2\,dt = \int_0^1 (4t^2-4t+1)\,dt$$
$$= \left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 = \frac{4}{3} - 2 + 1 = \frac{7}{3} - 2 = \frac{1}{3}.$$
$$\|g(t)\| = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} \quad \text{(all 3 answers fine).}$$

(d) [**10** points] Find the linear polynomial $k(t) = a + bt$ that best approximates the function $h(t) = t^2 - t$ on the interval $[0, 1]$ in the (continuous) least-squares sense.

This is done with a projection; we need $\mathrm{proj}_{\langle 1,t\rangle}(h(t))$. Best done with an ONB for $\langle 1, t\rangle$; now $f(t) \perp g(t)$, $\|f(t)\| = 1$, $\|g(t)\| = \frac{1}{\sqrt{3}}$. So $u_1 = f(t)$ and $u_2 = \frac{g(t)}{\|g(t)\|}$
$= \sqrt{3}\,g(t) = \sqrt{3}(2t-1)$ form an ONB!

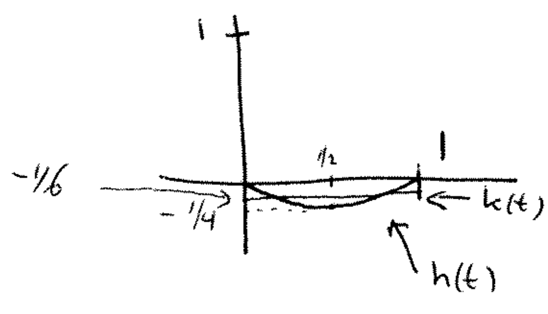
Then $\mathrm{proj}(h(t)) = \langle u_1, h(t)\rangle u_1 + \langle u_2, h(t)\rangle u_2$. But

$$\langle u_1, h(t)\rangle = \int_0^1 h(t)\,dt = \int_0^1 (t^2-t)\,dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

And
$$\langle u_2, h(t)\rangle = \int_0^1 \sqrt{3}(2t-1)(t^2-t)\,dt = \sqrt{3}\int_0^1 (2t^3 - 3t^2 + t)\,dt$$
$$= \sqrt{3}\left[\frac{2}{4}t^4 - t^3 + \frac{1}{2}t^2\right]_0^1 = \sqrt{3}\left(\frac{1}{2} - 1 + \frac{1}{2}\right) = 0.$$

So $k(t) = \mathrm{proj}(h(t)) = -\frac{1}{6}u_1 = -\frac{1}{6}$ (answer).

To understand it, picture:



$k(t)$ is the best linear function on $[0,1]$ approximating $h(t)$. Note that $h(t)$ is symmetric wrt. reflection in the midpoint $t = \frac{1}{2}$. So the same holds for $k(t)$, so $k(t)$ is horizontal. Finally, $\int_0^1 k(t)\,dt = \int_0^1 h(t)\,dt$ ... $k(t) = -1/6$ is best

4. [**25** points] Consider the matrix

$$A = \begin{bmatrix} -3 & 0 & 4 \\ 0 & -1 & 0 \\ -2 & 7 & 3 \end{bmatrix}.$$

(a) [**5** points] Find all real eigenvalues of A, with their algebraic multiplicities.

$$0 = \begin{vmatrix} -3-\lambda & 0 & 4 \\ 0 & -1-\lambda & 0 \\ -2 & 7 & 3-\lambda \end{vmatrix} \underset{\substack{\text{develop} \\ 2^{nd}\text{ row}}}{\overset{!}{=}} (-1-\lambda) \begin{vmatrix} -3-\lambda & 4 \\ -2 & 3-\lambda \end{vmatrix}$$

$$= -(\lambda+1)\left((\lambda-3)(\lambda+3)+8\right) = -(\lambda+1)(\lambda^2-9+8)$$

$$= -(\lambda+1)(\lambda^2-1) = -(\lambda+1)^2(\lambda-1).$$

Eigenvalues: -1, with alg. mult. 2;
1, with alg. mult. 1.

(b) [**5** points] For each eigenvalue of A, find a basis of the associated eigenspace. What are the geometric multiplicities of the eigenvalues of A?

$\underline{\lambda_1 = 1}$: $\mathrm{Ker}(A-I) = \mathrm{Ker}\begin{pmatrix} -4 & 0 & 4 \\ 0 & -2 & 0 \\ -2 & 7 & 2 \end{pmatrix}$. Need $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

with $\begin{pmatrix} -4 & 0 & 4 \\ 0 & -2 & 0 \\ -2 & 7 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$; then $-2x_2 = 0 \Rightarrow x_2 = 0$

and $-4x_1 + 4x_3 = 0 \Rightarrow x_3 = x_1$ & that's it!

So e.g. $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a basis. Geom. mult. 1 (is obvious

$\underline{\lambda_2 = -1}$: $\mathrm{Ker}(A+I) = \mathrm{Ker}\begin{pmatrix} -2 & 0 & 4 \\ 0 & 0 & 0 \\ -2 & 7 & 4 \end{pmatrix}$; $x_1 = 2x_3$, and $x_2 = 0$;

So $\vec{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ is a basis. Geom. mult. 1 (<u>not</u> obvious from the start).

(c) [**5 points**] Does there exist an eigenbasis for the matrix A? Motivate your answer.

No: only 2 independent eigenvectors.

Or: No, since sum of geom. mult.'s $= 1+1$
$= 2 < 3 =$ size of matrix $A = \dim \mathbb{R}^3$.

(d) [**5 points**] Is A diagonalizable? Motivate your answer.

No, since a matrix is diagonalizable if and only if there exists an eigenbasis for it.

(e) [**5 points**] Determine the eigenvalues of A^2, with their algebraic and geometric multiplicities.

One can compute A^2 (do it correctly!) but it's more fun to think a little:

in general, if $A\vec{v} = \lambda\vec{v}$, then $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v})$
$= \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$.

SO: if $\vec{v}$ is an eigenvector for A, with e.value λ, then $\vec{v}$ is an eigenvector for A^2, with e-value λ^2.

So: $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is an e.vector for A^2 with e-value $1^2 = 1$;

$\begin{pmatrix}2\\0\\1\end{pmatrix}$ is an e.vector for A^2 with e-value $(-1)^2 = 1$;

So the geom. mult. of **1** is at least 2.

But: $\det(A) = (-1)^2\, 1$ = product of e.values with alg. mult.

So $\det(A^2) = \det(A)\det(A) = 1^2 = 1 = 1\cdot 1\cdot(3^{rd}\text{ eigenvalue})$;

$\det(A) = 1$

so alg. mult of 1 is 3.

Finally, if geom. mult $= 3$, then $A^2 = I$. But easy to see that $A^2 \neq I$.
So geom. mult. is 2.