LINEAR ALGEBRA – SECOND MIDTERM EXAM SOLUTIONS

1 . The characteristic polynomial is:

$$p_A(\lambda) = \det \begin{pmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda - 2 \end{pmatrix} = (\lambda - 1)^2(\lambda - 2) \ .$$

So the eigenvalues of A are 1 and 2. The eigenspaces are:

$$E_1 = \ker \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} ,$$

$$E_2 = \ker \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} .$$

2 . We look for a line $y = mx + b$, where the data points correspond to (x, y). Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} .$$

Then the least squares solution is the vector $\begin{pmatrix} m \\ b \end{pmatrix}$ satisying:

$$A^T A \begin{pmatrix} m \\ b \end{pmatrix} = A^T \vec{y} \ .$$

$$A^T A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$A^T \vec{y} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

It is easy to see that $\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \end{pmatrix}$. The equation of the least squares line is therefore:

$$y = \frac{5}{2} x \ .$$

3 . (i) False. For example, if A is an invertible $n \times n$ matrix, $\operatorname{rref}(A) = I_n$, so $\det(\operatorname{rref}(A)) = 1$. But $\det(A)$ need not be 1.

$$\det\begin{pmatrix} 2 & 1 & 9 & 0 & 1 \\ 9 & 3 & 0 & 0 & 2 \\ 5 & 3 & 0 & 1 & 9 \\ 7 & 4 & 0 & 0 & 4 \\ 9 & 5 & 3 & 4 & 1 \end{pmatrix} = 3(-1)^{3+5}\begin{vmatrix} 2 & 1 & 0 & 1 \\ 0 & 3 & 0 & 2 \\ 5 & 3 & 1 & 9 \\ 7 & 4 & 0 & 4 \end{vmatrix} = 3 \cdot 1 (-1)^{3+3}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 7 & 4 & 4 \end{vmatrix} = 3\left(2 \cdot \begin{vmatrix} 3 & 2 \\ 4 & 4 \end{vmatrix} + 7\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\right)$$

$$= 3\left(2 \cdot (3 \cdot 4 - 4 \cdot 2) + 7(1 \cdot 2 - 3 \cdot 1)\right)$$

$$= 3(8-7) = 3$$

(ii) True.

(iii) True. Since

$$\|Ax\|^2 = \langle x, A^T Ax \rangle \ ,$$

if x is in the kernel of $A^T A$, then the right hand side is zero, so $Ax = 0$. Conversely, if x is in the kernel of A, it is also in the kernel of $A^T A$.

4 . (i) Just choose any orthonormal basis as column vectors, e.g.

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

(ii) No. For if A is 3×3 and skew-symmetric,

$$\det(A) = \det(A^T) = \det(-A) = (-1)^3 \det(A) = -\det(A) \ ,$$

so $\det(A) = 0$; A cannot be invertible.

5 . First find an orthonormal basis for the span of $\{1, x\}$. The function $w_1 = 1$ has unit length. Now

$$\langle x, 1 \rangle = \int_0^1 x dx = \frac{1}{2} \ ,$$

$P_1 = \text{projection of } x \text{ onto } 1 = \langle x, 1 \rangle 1 = 1/2$

and

$$\|x - 1/2\|^2 = \int_0^1 (x - 1/2)^2 dx = \frac{1}{3}(x - 1/2)^3\Big]_0^1 = \frac{1}{12} \ .$$

$W_2 = \dfrac{x - 1/2}{\|x - 1/2\|}$

So if we let $w_2 = 2\sqrt{3}(x - 1/2)$, then $\{w_1, w_2\}$ is an orthonormal set. Then the projection of h is given by:

$$\text{proj}(h) = \langle h, w_1 \rangle w_1 + \langle h, w_2 \rangle w_2 \ .$$

We compute:

$$\langle x^2, 1 \rangle = \int_0^1 x^2 dx = \frac{1}{3} \ .$$

$$\langle x^2, w_2 \rangle = 2\sqrt{3} \int_0^1 x^2(x - 1/2) dx = 2\sqrt{3}\left(\frac{x^4}{4} - \frac{x^3}{6}\right)\Big]_0^1 = \frac{\sqrt{3}}{6} \ .$$

So

$$\text{proj}(h) = \frac{1}{3} \cdot 1 + \frac{\sqrt{3}}{6} \cdot 2\sqrt{3}(x - 1/2) = x - \frac{1}{6} \ .$$