**LINEAR ALGEBRA**  
First Midterm Exam

JOHNS HOPKINS UNIVERSITY  
SPRING 2013

You have **50 MINUTES**.  
No calculators, books or notes allowed.

*Academic Honesty Certificate.* I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature: ___________________________  
Date: ________________

Name: ___________________________  
Section No: __________
(or TA's name)

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(1) (a) [15 points] Find all solutions to the system of equations:

\[
\begin{align*}
    x + y + 6z &= 8 \\
    2x + 3y + 16z &= 21
\end{align*}
\]

using Gaussian elimination. Is the system consistent? Why?

(b) [10 points] Does the system of equations:

\[
\begin{align*}
    2x + 12z &= 14 \\
    2y + 16z &= 18 \\
    x + 2y + 22z &= 25
\end{align*}
\]

have a unique solution? Justify your answer.
(2) [25 points] Let:

\[
A = \begin{bmatrix}
1 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 1 & 5 \\
1 & 1 & 1 & 15
\end{bmatrix}
\]

Can an equation:

\[A\vec{x} = \vec{b}\]

have infinitely many solutions while another equation:

\[A\vec{x} = \vec{c}\]

has none whatsoever? If no, explain why not. If yes, find vectors \(\vec{b}\) and \(\vec{c}\) in \(\mathbb{R}^4\) for which this is true.
(3) (a) [5 points] Compute the matrix products $BA$ and $AB$ where:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(b) [10 points] Does the matrix $A$ above have an inverse? If yes, compute it. If no, why not?
(3) (c) [10 points] Write down the matrix for the following linear transformation. (The origin is at the center of each drawing.) Explain how you reached your answer.

![Image of sloths](image_url)

[This must be how sloths look to one another!]

(4) (a) [5 points] What does it mean to say that vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent?

(b) [5 points] Are these vectors linearly independent? Justify your answer using determinants.

$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
(4) (c) [15 points] What is the dimension of the space spanned by the following vectors? Explain your approach and show your work.

\[ \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \]
(5) [20 bonus points] Find a basis for the image of the linear transformation:

\[
A = \begin{bmatrix}
a & a & b & a \\
a & a & b & 0 \\
a & b & a & b \\
a & b & a & 0 \\
\end{bmatrix}
\]

for any real numbers \(a\) and \(b\).

[Hint: The special cases \((a, b) = (0, 0)\) and \((a, b) = (0, 1)\) immediately show that the number of basis vectors will depend on the values \(a\) and \(b\) take, so carry out as much row reduction as possible without dividing by possibly vanishing numbers and break into cases at the last step.]