Name ....

Suggested Answers to Practice Exam 1
Math 201

1(a) *7pts.* Find all solutions to the given system of equations.

$$\begin{array}{rcrcrcrcl} x_1 & & & + & x_3 & + & x_4 & = & 0 \\ 2x_1 & + & x_2 & & & - & x_4 & = & 0 \\ x_1 & - & x_2 & + & 2x_3 & + & 2x_4 & = & 0 \\ x_1 & + & x_2 & + & 3x_3 & + & 6x_4 & = & 0 \end{array}$$

Soln We first write down the augmented matrix.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & -1 & 0 \\ 1 & -1 & 2 & 2 & 0 \\ 1 & 1 & 3 & 6 & 0 \end{bmatrix}$$

Do row operations, R2 - 2R1, R3 -R1 and R4 -R1.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 5 & 0 \end{bmatrix}$$

Do row operations, R3 + R2 and R4 - R2.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 4 & 8 & 0 \end{bmatrix}$$

Do row operation R4 + 4R3.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we apply back substitution to get

$$\begin{aligned} x_3 &= -2x_4 \\ x_2 - 2x_3 - 3x_4 &= 0 \\ \Longrightarrow x_2 &= -x_4 \\ x_1 &= -x_3 - x_4 \\ \Longrightarrow x_1 &= x_4 \end{aligned}$$

Solution is

$$\begin{cases} x_1 = x_4 \\ x_2 = -x_4 \\ x_3 = -2x_4 \\ x_4 \text{ is free} \end{cases}$$

1(b) *3pts.* Are the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 6 \end{bmatrix}$ linearly independent? Why or why not?

Soln: If we set up a homogeneous system for the vectors $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 6 \end{bmatrix}$ to check for linear independence we get the same system as in part (a). From solution to 1(a) we see that the system has infinitely many solutions. So they are *not linearly independent*.

2(a) *7pts.* Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. Compute the inverse of A, if it exists.

Soln:

$$\left[\begin{array}{rrr|rrr} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

Exchange rows R1 and R2.

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

R2 -2R1 and R3-2R1 give

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

R2 +3R3 and R1 -R3 imply,

$$\left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 0 & 3 & -1 \\ 0 & -1 & 0 & 1 & -8 & 3 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

Finally R1 +R2 and (-1)R3 give us

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -5 & 2 \\ 0 & 1 & 0 & -1 & 8 & -3 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{array}\right]$$

Therefore $A^{-1} = \begin{bmatrix} 1 & -5 & 2 \\ -1 & 8 & -3 \\ 0 & -2 & 1 \end{bmatrix}$.

2(b) *3pts.* Find all possible solutions for the system

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 2 \\ x_1 & + & x_2 & + & x_3 & = & 3 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & -1 \end{array}$$

Soln: In matrix form the above system is equivalent to $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Therefore $x = A^{-1} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -15 \\ 25 \\ -7 \end{bmatrix}$

(3) *8pts.* Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$.

3(a) *4pts.* Find all values of h such that $v = \begin{bmatrix} 1 \\ 8 \\ -1 \end{bmatrix}$ in the Span$\{v_1, v_2, v_3\}$.

Soln: For v to be in the Span$\{v_1, v_2, v_3\}$, the system $x_1v_1 + x_2v_2 + x_3v_3 = v$ should have a solution. So the following augmented system should be consistent.

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 \\ 2 & -1 & 1 & | & 8 \\ 1 & 2 & h & | & -1 \end{bmatrix}$$

R2-2R1 and R3 -R1 imply,

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 \\ 0 & -3 & 3 & | & 6 \\ 0 & 1 & h-2 & | & -2 \end{bmatrix}$$

R2/3 implies

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 \\ 0 & -1 & 1 & | & 2 \\ 0 & 1 & h-2 & | & -2 \end{bmatrix}$$

R3 + R2 then gives us

$$\begin{bmatrix} 1 & 1 & 2 & | & 1 \\ 0 & -1 & 1 & | & 2 \\ 0 & 0 & h-3 & | & 0 \end{bmatrix}$$

Now if $h-3 \neq 0$ then, the system will obviously have a unique solution. Also, if $h-3=0$ there will be a row of zeros and the system will have infinitely many solutions. Therefore, the v is in the $\text{Span}\{v_1, v_2, v_3\}$ for all values of h.

3(b) *4pts.* Find all values of h such that v_1, v_2 and v_3 linearly independent.

Soln: In order to check that v_1, v_2 and v_3 are linearly independent, we need to check that the system $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has only trivial solution. Now, from part (a) we will get the row echelon for the system as (only augemented column will change and in this case they are just a column of zeros) follows:

$$\left[\begin{array}{rrr|r} 1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & h-3 & 0 \end{array}\right]$$

Now, this system will have a unique solution only when $h-3 \neq 0$ i.e., $h \neq 3$. The vectors are hence linearly independent for all h other than 3.

(4) *12pts.* State True or False with justification. *3pts. each for the justification.*

(a) If A is a 2×2 matrix such that $A^2 = 0$ then $A = 0$.

Soln: False. Consider the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = 0$ but A is non-zero.

(b) If A, B and C are 2×2 invertible matrices then, $(AB)C$ is invertible.

Soln: True. Product of invertible matrices is invertible. Therefore, AB will be invertible. But C is also invertible hence, $(AB)C$ is also invertible. In fact its inverse is given by $C^{-1}B^{-1}A^{-1}$.

4(c) The vectors $\begin{bmatrix} 4 \\ -8 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -12 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \\ -1 \end{bmatrix}$, span $\mathbb{R}^3$.

Soln: False. Note that the vectors are scalar multiples of each other. Hence they can only span a line.

4(d) The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents reflection about a line.

Soln: False : Note that applying this matrix twice to the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ will give us a the vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. But if this was a reflection along a line, multiplying this matrix

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