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ETHICS PLEDGE: I agree to complete this exam without unauthorized assistance from any person or person's work, materials or device.

Your name (print): ________________________ Section: ________

Signature: ________________________________ Date: ________

INSTRUCTIONS: No books, no notes, no calculators! Write legibly, and show all relevant work—or expect to lose credit; no credit for guesses. Your solutions will be judged as answers to questions asked. They must conform to the writing standards to earn full credit.

If you need additional space to write your solutions to some problem, please use the back of the **preceding** page when possible. (It makes matters easier for everybody.)

1 [15]	2 [20]	3 [25]	4 [10]	5 [10]	6 [15]	7 [10]	T [105]

110.201 Prof. Zucker Exam 1: Oct. 16, 2007 Time: 85 minutes

Throughout, when A is a matrix, T_A denotes the associated linear transformation: $T_A(\vec{x}) = A\vec{x}$.

[15] 1. a) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Take as given that $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis (ordered) of $\mathbb{R}^3$. Determine the $\mathfrak{B}$-matrix of T_A (as an explicit 3×3 matrix).

b) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation, with $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Determine $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

[20] 2. Let $A = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & 5 \end{bmatrix}$.

a) Use the Gauss-Jordan elimination process to reduce A to reduced row-echelon form. In other words, determine $rref(A)$. **At each step of the process, specify which row operation you are using.**

Also, determine all (x_1, x_2, x_3, x_4) that satisfy the system of three equations:

$$2x_1 + x_2 + x_3 - 2x_4 = 3, \quad x_1 + x_2 + x_4 = 2, \quad x_1 + 2x_2 - x_3 + 5x_4 = 4.$$

b) Determine a basis for the kernel of A.

c) Determine the dimension of the image of A.

[5,5,5,10] 3. a) What is meant when one says that Y is a subspace of $\mathbb{R}^n$. That is, give the definition of "subspace of $\mathbb{R}^n$."

b) For Y a subspace of $\mathbb{R}^n$, give the definition of the **dimension** of Y.

c) Let $Y = \{\vec{x} \in \mathbb{R}^4 : 2x_1 + x_2 + x_3 - 2x_4 = 0\}$. Verify (using the definition in part a) that Y is a subspace of $\mathbb{R}^4$.

d) Determine a basis $\mathfrak{B}$ of $\mathbb{R}^4$ that contains a basis of the subspace Y from part c).

[10] 4. Determine the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates $\vec{e}_1$ in the counterclockwise direction through angle θ, and rotates $\vec{e}_2$ in the *clockwise* direction through angle θ?

[10] 5. Determine the dimension of

$$Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} \right\} \quad (\text{in } \mathbb{R}^4).$$

[15] 6. Let A and B be 2×2 matrices, with $\operatorname{rank} A = \operatorname{rank} B = 1$, and consider the product AB.

a) Give an example where $\operatorname{rank}(AB) = 1$, and an example where $\operatorname{rank}(AB) = 0$.

b) Show that $\operatorname{rank}(AB)$ cannot equal 2.

c) If we allow $\operatorname{rank} A = 2$ and $\operatorname{rank} B = 1$, can $\operatorname{rank}(AB) = 2$? **Explain.**

[10] 7. *True or false:* Let $T : \mathbb{R}^5 \to \mathbb{R}^6$ be a linear transformation, and let

$$\mathfrak{B} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5\}$$

be the standard basis of $\mathbb{R}^5$ (the domain). Then, $\{T(\vec{e}_1), T(\vec{e}_2), T(\vec{e}_3), T(\vec{e}_4), T(\vec{e}_5)\}$ must be linearly independent in $\mathbb{R}^6$. **Explain.**