THE JOHNS HOPKINS UNIVERSITY
Krieger School of Arts and Sciences
SOLUTIONS TO FIRST MIDTERM EXAM - FALL 2005
110.201 – LINEAR ALGEBRA

Instructor: Professor Carel Faber
Duration: 50 minutes   October 19, 2005

No calculators allowed **Total = 100 points**

1. [**20** points] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 8 \\ 2 & -2 & 3 \end{bmatrix}.$$

Check your answer.

Answer:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 3 & -2 & 8 & 0 & 1 & 0 \\ 2 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & -3 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & 0 & -1 \\ 0 & 1 & 0 & 7 & 1 & -5 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \to$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & 1 & -6 \\ 0 & 1 & 0 & 7 & 1 & -5 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right].$$

So

$$A^{-1} = \begin{bmatrix} 10 & 1 & -6 \\ 7 & 1 & -5 \\ -2 & 0 & 1 \end{bmatrix}.$$

Check:

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 8 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 10 & 1 & -6 \\ 7 & 1 & -5 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

and

$$\begin{bmatrix} 10 & 1 & -6 \\ 7 & 1 & -5 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 3 & -2 & 8 \\ 2 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Actually, it suffices to check one of the two, by Fact 2.4.9.

2. [**20** points] Solve the linear system

$$\left|\begin{array}{ccccccccccc} 2x_1 & + & 4x_2 & - & 2x_3 & - & 10x_4 & + & 2x_5 & = & 7 \\ 3x_1 & + & 6x_2 & + & 3x_3 & + & 3x_4 & - & 2x_5 & = & -4 \\ x_1 & + & 2x_2 & & & - & 2x_4 & & & = & 1 \end{array}\right|.$$

Check your answer.

Answer:

$$\begin{bmatrix} 2 & 4 & -2 & -10 & 2 & | & 7 \\ 3 & 6 & 3 & 3 & -2 & | & -4 \\ 1 & 2 & 0 & -2 & 0 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 & 0 & | & 1 \\ 3 & 6 & 3 & 3 & -2 & | & -4 \\ 2 & 4 & -2 & -10 & 2 & | & 7 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 3 & 9 & -2 & | & -7 \\ 0 & 0 & -2 & -6 & 2 & | & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 1 & 3 & 0 & | & -2 \\ 0 & 0 & -2 & -6 & 2 & | & 5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 2 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 1 & 3 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & 2 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 & 0 & | & 1 \\ 0 & 0 & 1 & 3 & 0 & | & -2 \\ 0 & 0 & 0 & 0 & 1 & | & \frac{1}{2} \end{bmatrix}.$$

Thus $x_2 = s$ and $x_4 = t$ (the free variables) and $x_5 = \frac{1}{2}$, $x_3 = -3x_4 - 2 = -2 - 3t$, and $x_1 = -2x_2 + 2x_4 + 1 = 1 - 2s + 2t$.

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 - 2s + 2t \\ s \\ -2 - 3t \\ t \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ \frac{1}{2} \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

is the solution of the linear system (where s and t are arbitrary real numbers).

Check:

$$2(1 - 2s + 2t) + 4s - 2(-2 - 3t) - 10t + 2(\tfrac{1}{2}) = 2 - 4s + 4t + 4s + 4 + 6t - 10t + 1 = 7,$$
$$3(1 - 2s + 2t) + 6s + 3(-2 - 3t) + 3t - 2(\tfrac{1}{2}) = 3 - 6s + 6t + 6s - 6 - 9t + 3t - 1 = -4,$$

and

$$(1 - 2s + 2t) + 2s - 2t = 1 - 2s + 2t + 2s - 2t = 1.$$

3. [**20** points] Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with matrix

$$A = \begin{bmatrix} 2 & -3 & 12 & 17 \\ 3 & 2 & 5 & 6 \\ 1 & 4 & -5 & -8 \end{bmatrix}$$

and let $U : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with matrix

$$B = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is given that $B = \text{rref}(A)$.

(a) [**5** points] Determine $T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$.

(b) [**5** points] Determine a basis for $\ker(U)$. Show your work.

(c) [**5** points] Determine a basis for $\ker(T)$. Show your work.

(d) [**5** points] Determine a basis for $\text{im}(T)$. Show your work.

Answer:

(a) $T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ 11 \\ -13 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. (Thus $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ is in $\ker(T)$).

(b) Solving $B\vec{x} = \vec{0}$ immediately gives $x_3 = s$, $x_4 = t$, $x_1 = -3x_3 - 4x_4 = -3s - 4t$, and $x_2 = 2x_3 + 3x_4 = 2s + 3t$. Thus $\vec{x} = \begin{bmatrix} -3s - 4t \\ 2s + 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ is the solution. It follows immediately that

$$\left(\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right)$$

is a basis for $\ker(U)$.

(c) Since $B = \text{rref}(A)$, we have that $\ker(A) = \ker(B)$, thus $\ker(T) = \ker(U)$. So we can take the same basis as in (b).

(d) We know that a basis for $\text{im}(T) = \text{im}(A)$ is given by the column vectors of A corresponding to the column vectors of B that contain a leading 1. So

$$\left(\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \right)$$

is a basis for $\text{im}(T)$.

4. [**20** points] Let P_2 be the linear space of all polynomials of degree ≤ 2. It is a subspace of $F(\mathbb{R}, \mathbb{R})$. Consider the following elements of P_2:

$$f_1 = 1 - 2x + x^2, \qquad f_2 = 2 - 3x + 5x^2, \qquad f_3 = x - 2x^2, \qquad f_4 = -x^2.$$

(a) [**5** points] Prove that f_1, f_2, f_3, f_4 are linearly dependent elements of P_2.

(b) [**5** points] Prove that f_1, f_2, f_3, f_4 span P_2.

(c) [**5** points] Prove that $\mathcal{B} = (f_1, f_3, f_4)$ is a basis of P_2.

(d) [**5** points] Find the $\mathcal{B}$-coordinate vector of f_2.

Answer:

(a) We know that $\dim(P_2) = 3$. That means that any four elements of P_2 are linearly dependent. Alternatively, $2f_1 - f_2 + f_3 - 5f_4 = 0$ is a nontrivial linear relation between these four elements.

(b) We see that $x^2 = -f_4$, $x = f_3 - 2f_4$, and $1 = f_1 + 2f_3 - 3f_4$. Thus any polynomial of degree ≤ 2 can be written as a linear combination of f_1, f_3, f_4. So these three elements span P_2 already. So f_1, f_2, f_3, f_4 span P_2 as well.

(c) In (b) we already saw that f_1, f_3, f_4 span P_2. But it is easy to see that they are linearly independent as well. We can do this by explicit calculation (check that there is no nontrivial linear relation between these three elements) or we can use the fact that three elements in a linear space of dimension 3 that span the linear space are automatically linearly independent. (The analogue of Fact 3.3.4.d.)

(d) From (a) we have $f_2 = 2f_1 + f_3 - 5f_4$, thus $[f_2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$ is the $\mathcal{B}$-coordinate vector of f_2.

5. [**20** points] Here we consider linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(a) [**4** points] Let S be the reflection in the line $y = -x$. Determine the standard matrix of S.

(b) [**4** points] Let T be the reflection in the line $y = x\sqrt{3}$. Determine the standard matrix of T.
(Note that the angle between $\vec{a} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ equals $\pi/3$.)

(c) [**4** points] Determine the standard matrix of the composite transformation ST.

(d) [**4** points] Prove that ST is a rotation and find the angle of rotation (in the counterclockwise direction).

(e) [**4** points] Is TS the inverse of ST ? Explain your answer as well as you can.

Answer:

(a) $S\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ yields the first column and $S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ yields the second column. The matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

(b) $T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2}\sqrt{3} \end{bmatrix}$ yields the first column and $T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} \\ \sin\frac{\pi}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} \\ \frac{1}{2} \end{bmatrix}$ yields the second column. The matrix is $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$.

(c) It is

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}.$$

(d) Geometrically, one sees that first reflecting in $y = x\sqrt{3}$ and then in $y = -x$ is the same as a counterclockwise rotation over an angle that is twice as large as the angle between $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The angle between the two vectors is $\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$, so the angle of rotation is $\frac{5\pi}{6}$.
Alternatively, one recognizes that the matrix in (c) is of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for $\theta = \frac{5\pi}{6}$. Thus ST is a counterclockwise rotation over $\frac{5\pi}{6}$.

(e) The answer is yes. One can see this by computing the matrix of TS as in (c) and checking that that matrix is the inverse of the matrix for ST. But this doesn't provide so much explanation. One explanation is as follows: if S and T are invertible linear transformations (as is the case here), then the inverse of ST is $T^{-1}S^{-1}$. This holds in general. But in our case, T and S are reflections. A reflection equals its own inverse! So $T^{-1}S^{-1} = TS$ is the inverse of ST. Another explanation is more geometric: just as in (d), TS is a rotation, but the angle of rotation is opposite to the angle of rotation of ST (check this). Thus TS is the inverse of ST.