LINEAR ALGEBRA – FIRST MIDTERM EXAM SOLUTIONS

1 . *Solution.* The augmented matrix for this system is:

$$\begin{pmatrix} 1 & 1 & -1 & \vdots & 1 \\ -5 & 1 & 1 & \vdots & -7 \\ 1 & -5 & 3 & \vdots & 3 \end{pmatrix}.$$

Now compute the reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & -1/3 & \vdots & 4/3 \\ 0 & 1 & -2/3 & \vdots & -1/3 \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}.$$

This says that a particular solution is $z = 0$, $x = 4/3$, and $y = -1/3$. On the other hand, any other choice of z also gives a solution, so the solution is not unique.

2 . *Solution.* Compute the reduced row echelon form of this matrix to get:

$$\begin{pmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The leading variables are x_1, x_3 and x_5, so a basis for the image consists of the corresponding columns of the original matrix:

$$\text{Basis for the image:} \quad \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}$$

For the kernel, look at the nonleading variables x_2 and x_4. Set $x_2 = 1$, $x_4 = 0$ and solve the homogeneous equation to get $x_1 = 1$ and $x_3 = x_5 = 0$. Now set $x_2 = 0$, $x_4 = 1$ and solve the homogeneous equation to get $x_1 = -2$, $x_3 = -1$ and $x_5 = 0$. Hence,

$$\text{Basis for the kernel:} \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

3 . (i) True. The reduced row echelon form is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Three leading ones means the rank is three.

(ii) False. Take $\vec{u} = \vec{v}$ and $\vec{w}$ independent of $\vec{u}$, for example.

(iii) True. Note that by linearity, $\ker B \subset \ker AB$ is always true. So we must show that $\ker AB \subset \ker B$. If $AB\vec{x} = \vec{0}$, then $B\vec{x} \in \ker A$. But if A is invertible, $\ker A = \{\vec{0}\}$, so $\vec{x} \in \ker B$.

(iv) False. Linearity only holds if $\vec{b} = 0$.

4 . *Solution.* To prove linearity, we must show: (i) $T(\lambda\vec{x}) = \lambda T(\vec{x})$ for all $\lambda \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^3$, and (ii) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbb{R}^3$. For (i),

$$\begin{aligned} T(\lambda\vec{x}) &= (\lambda\vec{x} \cdot \vec{u}_1)\,\vec{u}_2 + (\lambda\vec{x} \cdot \vec{u}_2)\,\vec{u}_1 \\ &= \lambda\,(\vec{x} \cdot \vec{u}_1)\,\vec{u}_2 + \lambda\,(\vec{x} \cdot \vec{u}_2)\,\vec{u}_1 \\ &= \lambda T(\vec{x}) \end{aligned}$$

For (ii),

$$\begin{aligned} T(\vec{x} + \vec{y}) &= (\{\vec{x} + \vec{y}\} \cdot \vec{u}_1)\,\vec{u}_2 + (\{\vec{x} + \vec{y}\} \cdot \vec{u}_2)\,\vec{u}_1 \\ &= (\vec{x} \cdot \vec{u}_1)\,\vec{u}_2 + (\vec{y} \cdot \vec{u}_1)\,\vec{u}_2 \\ &\quad + (\vec{x} \cdot \vec{u}_2)\,\vec{u}_1 + (\vec{y} \cdot \vec{u}_2)\,\vec{u}_1 \\ &= T(\vec{x}) + T(\vec{y}) \end{aligned}$$

To compute the matrix, we evaluate:

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \vec{u}_2 - \vec{u}_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= \vec{u}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Hence, the matrix of T is:

$$[T] = \begin{pmatrix} -2 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

5 . Let $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Then $S^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, and

$$[T]_{\mathcal{B}} = S^{-1}[T]_{\mathcal{E}}S = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ -4 & -7 \end{pmatrix}$$