8. Lecture 8: 3.1 Image and Kernel of a Linear Transformation

If \( T: X \to Y \) is a transformation then the set \( X \) is called the **domain** of \( T \). The set \( \text{Im}(T) \) of all images \( T(x) \) when \( x \) varies over all points in the domain is called the **image** of \( T \), or sometimes the **range**. Note that the image need not be all of the **target space** \( Y \).

\( T \) is said to be **onto** if each \( y \in Y \) is the image \( T(x) \) of at least one \( x \in X \). \( T \) is said to be **one-to-one** if each \( y \in Y \) is the image of at most one \( x \in X \). \( T \) is called **invertible** if its one-to-one and onto.

**Ex 7** Define \( T: \mathbb{R}^2 \to \mathbb{R}^3 \) by \( T(x) = Ax \), where \( A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \). Is \( T \) onto? What is the image?

**Sol** The image of \( T \) is all combinations of the column vectors of \( A \)

\[
T(x) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} x_2
\]

for any \( x_1 \) and \( x_2 \). The image is the plane 'spanned' by the two column vectors.

Given vectors \( v_1, \ldots, v_k \) and scalars \( \lambda_1, \ldots, \lambda_k \), the vector

\[
w = \lambda_1 v_1 + \cdots + \lambda_k v_k
\]

is called a **linear combination** of the vectors \( v_1, \ldots, v_k \), with weights \( \lambda_1, \ldots, \lambda_k \).

The set of all linear combinations of a \( v_1, \ldots, v_n \) is called the **span** of \( v_1, \ldots, v_n \) and is denoted by \( \text{Span}(v_1, \ldots, v_n) \). The set \( \{v_1, \ldots, v_n\} \) span (is a spanning set for) \( V \) if every vector in \( V \) can be written as a linear combination of \( v_1, \ldots, v_n \).

**Th** The image of a linear transformation \( T(x) = Ax \) is the span of the column vectors of \( A \).

**Pf**

\[
T(x) = Ax = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 v_1 + \cdots + x_n v_n
\]

The image of a linear transformation \( T(x) = Ax \) is also called the column space of \( A \), \( \text{Col}(A) \).

A **subspace** \( W \) of \( \mathbb{R}^n \) is a subset which is closed under addition and scalar multiplication:
(a) \( 0 \in W \), (b) \( u \in W \) and \( v \in W \) then \( u + v \in W \), (c) \( w \in W \) and \( k \) is a scalar then \( kw \in W \).

**Ex** A plane \( ax_1 + bx_2 + cx_3 = 0 \) going through the origin in space is a subspace of \( \mathbb{R}^3 \).

**Th** The image of a linear transformation \( T(x) = Ax \), from \( \mathbb{R}^n \to \mathbb{R}^m \) is a subspace of \( \mathbb{R}^m \).

**Pf** For a proof see the proof of Theorem 3.1.4 in the textbook.

Alternatively it follows from the previous theorem and the following theorem:

**Th** If \( v_1, \ldots, v_n \in \mathbb{R}^m \) then \( \text{Span}(v_1, \ldots, v_n) \) is a subspace of \( \mathbb{R}^m \).

**Pf** (b) follows from that sums of linear combinations are linear combination. In fact let \( W = \text{Span}(v_1, \ldots, v_n) \). Then if \( u = c_1 v_1 + \cdots + c_n v_n \in W \) and \( w = d_1 v_1 + \cdots + d_n v_n \in W \) it follows that \( u + w = (c_1 + d_1) v_1 + \cdots + (c_n + d_n) v_n \in W \) since it is also a linear combination.
The kernel, \( \text{Ker}(T) \), of a linear transformation \( T: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is the set of all \( x \) in the domain such that \( T(x) = 0 \). It is a proper subset of the domain \( \mathbb{R}^n \) unless \( T \) is the zero map.

**Ex** Let \( T: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) by \( T(x) = Ax \), where \( A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \). Is \( T \) one-to-one? What is the kernel?

**Sol** \( Ax = 0 \) has nontrivial solutions since there are more variables than equations. Hence there are infinitely many points such that \( T(x) = 0 \) so \( T \) is not one-to-one. Explicitly

\[
\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}, \quad \iff \begin{align*}
    x_1 &= 2x_3 \\
    x_2 &= -x_3 \\
    x_3 &= \text{free}
\end{align*}
\]

The kernel is hence the subspace spanned by the line \( x = \begin{bmatrix} -1 \\ 1 \end{bmatrix} t \), for any parameter \( t \).

The kernel of a linear transformation \( T(x) = Ax \) is also called the null space of \( A \), \( \text{Nul}(A) \).

**Th** The kernel of a linear transformation \( T(x) = Ax \), from \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a subspace of \( \mathbb{R}^n \).

**Pf** We must verify the three properties (a), (b), (c) in the definition of subspace.

(a) \( 0 \in \text{Nul}(A) \) since \( A0 = 0 \).

(b) If \( u, v \in \text{Nul}(A) \), show that \( u + v \in \text{Nul}(A) \).

(c) If \( u \in \text{Nul}(A) \), show that \( \lambda u \in \text{Nul}(A) \).

**Ex 1** Find an explicit description of \( \text{Nul}(A) \) where \( A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \).

**Sol** Row reduction to solve \( Ax = 0 \):

\[
\begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \sim (1) \begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \sim (2) -6(1) \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \sim (1) -2(2) \begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix} \sim (1) \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \sim (1) /3 \begin{bmatrix} 1 & 2 & 2 & 1 & 3 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix} \sim (1) -2(2) \begin{bmatrix} 1 & 2 & 0 & 13 & 33 \\ 0 & 0 & 1 & -6 & -15 \end{bmatrix}
\]

Hence \( Ax = 0 \iff \begin{align*}
    x_1 + 2x_2 + 13x_4 + 33x_5 &= 0 \\
    x_3 - 6x_4 - 15x_5 &= 0
\end{align*} \). \( x_2, x_4, x_5 \) are free so the sol. is

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 \\
    x_4 \\
    x_5
\end{bmatrix} = \begin{bmatrix}
    -2x_2 - 13x_4 - 33x_5 \\
    -6x_4 + 15x_5 \\
    x_4 \\
    x_5
\end{bmatrix} = x_2 \begin{bmatrix}
    -2 \\
    1 \\
    0 \\
    0
\end{bmatrix} + x_4 \begin{bmatrix}
    -13 \\
    6 \\
    1 \\
    0
\end{bmatrix} + x_5 \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    1
\end{bmatrix}
\]

Hence \( \text{Nul}(A) = \text{Span}\{u, v, w\} \), is the span of the three vectors \( u, v, w \) above.
We always have that $0 \in \text{Ker}(A)$. When is $\text{Ker}(A) = \{0\}$?

**Th** (a) If $A$ is $m \times n$ then $\text{Ker}(A) = \{0\}$ if and only if $\text{rank}(A) = n$.
(b) If $A$ is $m \times n$ and $\text{Ker}(A) = \{0\}$ then $m \leq n$.
(c) If $A$ is $n \times n$ then $\text{Ker}(A) = \{0\}$ if and only if $A$ is invertible.

**Th** For an $n \times n$ matrix $A$ the following statements are equivalent:
(i) $A$ is invertible
(ii) $Ax = b$ has a unique solution $x$ for all $b$.
(iii) $\text{Rref}(A) = I$.
(iv) $\text{rank}(A) = n$.
(v) $\text{Im}(A) = \mathbb{R}^n$.
(vi) $\text{Ker}(A) = 0$
Summary and Questions

If $T: X \to Y$ is a transformation then the set $X$ is called the domain of $T$. The set $\text{Im}(T)$ of all images $T(x)$ when $x$ varies over all points in the domain is called the image of $T$, or sometimes the range. Note that the image need not be all of the target space $Y$.

$T$ is said to be onto if each $y \in Y$ is the image $T(x)$ of at least one $x \in X$.
$T$ is said to be one-to-one if each $y \in Y$ is the image of at most one $x \in X$.
$T$ is called invertible if its one-to-one and onto.

Given vectors $v_1, \ldots, v_k$ and scalars $\lambda_1, \ldots, \lambda_k$, the vector
$$w = \lambda_1 v_1 + \cdots + \lambda_k v_k$$
is called a linear combination of the vectors $v_1, \ldots, v_k$ (with weights $\lambda_1, \ldots, \lambda_k$).

The set of all linear combinations of a $v_1, \ldots, v_n$ is called the span of $v_1, \ldots, v_n$ and is denoted by $\text{Span}(v_1, \ldots, v_n)$. The set $\{v_1, \ldots, v_n\}$ is said to span $W$ if $W = \text{Span}(v_1, \ldots, v_n)$.

The image of a linear transformation $T(x) = Ax$ is the span of the column vectors of $A$.

The kernel, $\text{Ker}(T)$, of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is the set of all $x$ in the domain such that $T(x) = 0$. It is a proper subset of the domain $\mathbb{R}^n$ unless $T$ is the zero map.

A subspace $W$ of $\mathbb{R}^n$ is a subset which is closed under addition and scalar multiplication: (a) $0 \in W$, (b) $u + v \in W$ and $v \in W$ then $u + v \in W$, (c) $w \in W$ and $k$ is a scalar then $kw \in W$.

Ex A plane $ax_1 + bx_2 + cx_3 = 0$ going through the origin in space is a subspace of $\mathbb{R}^3$.

Th If $v_1, \ldots, v_n \in \mathbb{R}^m$ then $\text{Span}(v_1, \ldots, v_n)$ is a subspace of $\mathbb{R}^m$.

Th The image of a linear transformation $T(x) = Ax$, from $\mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$.

Th The kernel of a linear transformation $T(x) = Ax$, from $\mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^n$.

Question Which of the following are subspaces: (a) the plane $x_1 + 2x_2 - 4x_3 = 1$, (b) The span of the vectors $(1, 2, 4)$ and $(2, 4, 8)$? (c) The Kernel of the matrix corresponding to rotation by 90 degrees counterclockwise? (d) The circle $x_1^2 + x_2^2 = 1$. (e) The ball $x^2 + y^2 + z^2 \leq 1$.

Question Which of the following transformations have a nontrivial kernel (i.e. containing more than just 0)? (a) Rotation by $\pi/2$ counterclockwise, (b) Projection of the plane onto the $x$ axis. (c) Reflection in the $x$ axis.