27. Lecture 27: 7.5 Complex Eigenvalues

Complex Numbers

The complex plane $\mathbb{C}$ is just the real plane $\mathbb{R}^2$ with an additional structure given by multiplication defined as follows. The multiplication of two vectors should be linear in each argument and commutative and the result should be a vector in the plane. If $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ then we want

$$\mathbf{e}_1\mathbf{e}_1 = \mathbf{e}_1, \qquad \mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2, \qquad \mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_1.$$

To simplify notation one introduces the notation i for $\mathbf{e}_2$ and calls a multiple of it an imaginary numbers whereas a multiple of $\mathbf{e}_1$ is called a real numbers so

$$z = a + i\,b \quad \text{denotes the vector} \quad a\,\mathbf{e}_1 + b\,\mathbf{e}_2.$$

With this construction we can hence find a square root of a negative number

$$i^2 = i\,i = -1.$$

Moreover, we solve any polynomial equation within the complex numbers.
If $z = a + ib$ then the **complex conjugate** $\overline{z} = a - ib$ is the reflection in the real axis.
The multiplication of complex numbers satisfy

$$(a + i\,b)(c + i\,d) = ac - bd + i\,(ad + bc),$$

which is perhaps not so illuminating. However, the **polar form** of a complex number

$$z = a + i\,b = r(\cos\theta + i\,\sin\theta), \qquad \text{where} \quad r = \sqrt{a^2 + b^2},$$

leads to more insight as we shall see. Here one calls $|z| = r$ the absolute value of z and $\arg(z) = \theta$ the argument of z. Let w be another complex number in polar form

$$w = c + i\,d = \rho(\cos\phi + i\,\sin\phi), \qquad \text{where} \quad \rho = \sqrt{c^2 + d^2}.$$

If we multiply their polar forms we get

$$zw = r\rho(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = r\rho\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i\,(\sin\theta\cos\phi + \cos\theta\sin\phi)\big).$$

If we use some trigonometric identities this simplifies to

$$zw = r\rho\big(\cos(\theta + \phi) + i\,\sin(\theta + \phi)\big).$$

Hence

$$|zw| = |z|\,|w|, \qquad \arg(z\,w) = \arg(z) + \arg(w).$$

Since the arguments add as for the exponential function; $e^x e^y = e^{x+y}$, it is natural to extend the definition of the exponential function to complex arguments and in particular define

$$e^{i\theta} = \cos\theta + i\,\sin\theta.$$

This should simply be thought as a notation reminding us that it satisfies $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$.

Ex 1 Write $z = 1 + i$ and $\overline{z} = 1 - i$ in polar form $z = r(\cos\theta + i\,\sin\theta)$.
Sol $|z| = r = \sqrt{1^2 + 1^2} = \sqrt{2}$ so $z = \sqrt{2}\big(1/\sqrt{2} + i/\sqrt{2}\big) = \sqrt{2}\,(\cos\theta + i\,\sin\theta)$, for some θ. The unique $0 \le \theta < 2\pi$ satisfying $\cos\theta = 1/\sqrt{2}$ and $\sin\theta = 1/\sqrt{2}$ is $\arg(z) = \theta = \pi/4$. Hence $z = \sqrt{2}\big(\cos(\pi/4) + i\sin(\pi/4)\big)$ and $\overline{z} = \sqrt{2}\big(\cos(\pi/4) - i\sin(\pi/4)\big)$.

Fundamental theorem of algebra Any polynomial of degree n with complex coefficients can be written as $p(\lambda) = k(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)$, for some complex k and $\lambda_1, \ldots, \lambda_n$.

Complex Eigenvalues

Ex 1 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Sol This is the matrix for a rotation with scaling: $A = \sqrt{2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta = \pi/4$ and can not have any real eigenvectors unless the rotation a multiple of π.

The complex eigenvalues are solution of:
$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1^2 = (1-\lambda-i)(1-\lambda+i) = 0,$
i.e. $\lambda = \lambda_1 = 1 + i$, or $\lambda = \lambda_2 = 1 - i$. The eigenvectors are solutions to:

$$(A - \lambda_1 I)\mathbf{v}_1 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Leftrightarrow\quad \begin{matrix} -ix_1 + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{matrix} \quad\Leftrightarrow\quad \mathbf{v}_1 = \alpha\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{v}_2 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad\Leftrightarrow\quad \begin{matrix} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{matrix} \quad\Leftrightarrow\quad \mathbf{v}_2 = \beta\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Even though in many applications we are looking for real solutions the complex solutions can still be helpful on the way towards a final answer as we shall see.

We can now complex diagonalize A. If $S = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ then

$$AS = \begin{bmatrix} | & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = SD$$

Hence $A = SDS^{-1}$, where $S^{-1} = \begin{bmatrix} i/2 & i/2 \\ -1/2 & 1/2 \end{bmatrix}$

Ex 2 Find A^k, where $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Sol We have $A^k = (SDS^{-1})^k = SDS^{-1}\cdots SDS^{-1} = SD^kS^{-1}$, where $D^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$. We have $\lambda_1 = \sqrt{2}\big(\cos(\pi/4) + i\sin(\pi/4)\big)$ so $\lambda_1^k = \sqrt{2}\big(\cos(k\pi/4) + i\sin(k\pi/4)\big)$ and $\lambda_2^k = \sqrt{2}\big(\cos(k\pi/4) - i\sin(k\pi/4)\big)$. Even though S, S^{-1} and D^k are complex we know that the end result A^k is real and the complex diagonalization gives a way to calculate it.

Summary

Complex plane is $\mathbb{R}^2$ with an extra multiplicative structure

$$z = a + i\,b \quad \text{where} \quad i^2 = -1$$

The polar representation of a complex number

$$z = a + i\,b = r(\cos\theta + i\,\sin\theta), \qquad \text{where} \quad r = \sqrt{a^2 + b^2},$$

Ex Write $1 + i = \sqrt{2}\big(\cos(\pi/4) + i\sin(\pi/4)\big)$ and $1 - i = \sqrt{2}\big(\cos(\pi/4) - i\sin(\pi/4)\big)$.
Let w be another complex number in polar form

$$w = c + i\,d = \rho(\cos\phi + i\,\sin\phi), \qquad \text{where} \quad \rho = \sqrt{c^2 + d^2},$$

Then

$$zw = r\rho\big(\cos(\theta + \phi) + i\,\sin(\theta + \phi)\big)$$

Complex Eigenvalues

Ex 1 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and use it to calculate A^k.

Sol This is the matrix for a rotation with scaling: $A = \sqrt{2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta = \pi/4$ and can not have any real eigenvectors unless the rotation a multiple of π.

The complex eigenvalues are solution of:
$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1^2 = (1 - \lambda - i)(1 - \lambda + i) = 0,$
i.e. $\lambda = \lambda_1 = 1 + i$, or $\lambda = \lambda_2 = 1 - i$. The eigenvectors are solutions to:

$$(A - \lambda_1 I)\mathbf{v}_1 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{c} -ix_1 + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{array} \quad \Leftrightarrow \quad \mathbf{v}_1 = \alpha\begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{v}_2 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{array}{c} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{array} \quad \Leftrightarrow \quad \mathbf{v}_2 = \beta\begin{bmatrix} i \\ 1 \end{bmatrix}$$

Even though in many applications we are looking for real solutions the complex solutions can still be helpful on the way towards a final answer as we shall see.

We can now complex diagonalize A. If $S = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ then

$$AS = \begin{bmatrix} | & | \\ A\mathbf{v}_1 & A\mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{v}_1 & \mathbf{v}_2 \\ | & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = SD$$

Hence $A = SDS^{-1}$.
We have $A^k = (SDS^{-1})^k = SDS^{-1}\cdots SDS^{-1} = SD^kS^{-1}$, where $D^k = \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}$.
We have $\lambda_1 = \sqrt{2}\big(\cos(\pi/4) + i\sin(\pi/4)\big)$ so $\lambda_1^k = \sqrt{2}\big(\cos(k\pi/4) + i\sin(k\pi/4)\big)$ and $\lambda_2^k = \sqrt{2}\big(\cos(k\pi/4) - i\sin(k\pi/4)\big)$. Even though S, S^{-1} and D^k are complex we know that the end result A^k is real and the complex diagonalization gives a way to calculate it.