26. Lecture 26: 7.4 Discrete Dynamical Systems

Transition Matrices

Ex $A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$ is a transition matrix for a mini-web with 3 pages, e.g. the entry in the first column and second row tell us that 20% of those on Page 1 will move to Page 2.

The evolution is given by $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $k = 1, ...$ Suppose we start at $\mathbf{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

What happens as $k \to \infty$?

It turns out that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0.5$ and $\lambda_3 = 0.2$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

We can write

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3, \qquad \text{where} \quad c_1 = \frac{1}{20}, \ c_2 = \frac{2}{45}, \ c_3 = \frac{1}{36}.$$

Since $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ it follows that $A^k\mathbf{v}_i = \lambda_i^k\mathbf{v}_i$ so

$$A^k\mathbf{x}_0 = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + c_3\lambda_3^k\mathbf{v}_3 = \frac{1}{20}1^k \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix} + \frac{2}{45}(0.5)^k \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{36}(0.2)^k \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix} \to \frac{1}{20} \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix},$$

as $k \to \infty$.

A **distribution vector** is a vector with all components positive or 0 and adding up to 1. A **transition matrix** is a matrix in which each column vector is a distribution vector.

Th Let A be an $n\times n$ transition matrix. Then there is exactly one eigenvector $\mathbf{x}_{equ}$, called the **equilibrium distribution**, with eigenvalue 1. All other eigenvalues are $|\lambda| < 1$.
Pf Since the determinant of the transpose is the same as the determinant of the matrix it follows that the characteristic polynomial for A is the same as for A^T, so the eigenvalues of A are the same as the eigenvalues A^T and in fact the geometric multiplicities are the same because the ranks are the same. It is easy to see that the vector $\mathbf{w} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector to A^T with eigenvalue 1. In fact since the entries of each row add up to 1 the dot product of the rows with this vector is also 1. Now, suppose that $\mathbf{x}$ is another eigenvector $A^T\mathbf{x} = \lambda\mathbf{x}$. Let x_i be the largest component so $x_i \geq x_k$ for all k with strict inequality for some k, and let $\mathbf{r}_i$ be the i-th row of A. Then $\lambda x_i = \mathbf{r}_i\cdot\mathbf{x} = r_{i1}x_1 + \cdots + r_{in}x_n < r_{i1}x_i + \cdots + r_{in}x_i = x_i$ so $\lambda < 1$. A similar argument shows that $|\lambda| \leq 1$ and that $\lambda \neq -1$. For later use we also note that:
Lem Let $\mathbf{w} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$. Then $A^T\mathbf{w} = \mathbf{w}$ and $(A\mathbf{x})\cdot\mathbf{w} = \mathbf{x}\cdot\mathbf{w}$.
Moreover, if $\mathbf{v}$ is an eigenvector of A with eigenvalue $\lambda \neq 1$ then $\mathbf{w}\cdot\mathbf{v} = 0$.
Pf Since $A^T\mathbf{w} = \mathbf{w}$ we have $\mathbf{w}\cdot\mathbf{v} = (A^T\mathbf{w})\cdot\mathbf{v} = \mathbf{w}\cdot A\mathbf{v} = \lambda\mathbf{w}\cdot\mathbf{v}$, which proves that $\mathbf{w}\cdot\mathbf{v} = 0$.

Let $\mathbf{v}_k$ be the eigenvectors of A, with eigenvalues $\lambda_1 = 1$ and $|\lambda_k| < 1$ for $k > 1$. It follows that the system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ which is initially is in the state $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ will after k times steps be in the state

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

and as $k \to \infty$

$$\mathbf{x}_k \to c_1\mathbf{v}_1 = \mathbf{x}_{equ}.$$

Rem Note that we only have to calculate $\mathbf{v}_1$ in order to calculate $\mathbf{x}_{equ}$. This is because it follows from the lemma that $\mathbf{x}_0\cdot\mathbf{w} = c_1\mathbf{v}_1\cdot\mathbf{w}$, so c_1 is determined directly from $\mathbf{x}_0$ and $\mathbf{v}_1$.

Ex Denote the owl and rat population at time k by $\mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix}$. Suppose

$$\begin{aligned} O_{k+1} &= 0.5\,O_k + 0.4\,R_k \\ R_{k+1} &= -p\,O_k + 1.1\,R_k \end{aligned}$$

where $p = 0.104$, or $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where $A = \begin{bmatrix} 0.5 & 0.4 \\ -.104 & 1.1 \end{bmatrix}$. The eigenvalues for the matrix A are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$ and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. An initial $\mathbf{x}_0$ can be written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then for $k \geq 0$

$$\mathbf{x}_k = c_1 A^k \mathbf{v}_1 + c_2 A^k \mathbf{v}_2 = c_1 \lambda_1^k \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 = c_1 (1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2 (0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

As k becomes large the first state will dominate and the other will go to $\mathbf{0}$ unless the initial conditions are such that $c_1 = 0$ in which case the whole solution goes to $\mathbf{0}$.

Summary

Ex $A = \begin{bmatrix} 0.7 & 0.1 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}$ is a transition matrix for a mini-web with 3 pages, e.g. the entry in the first column and second row tell us that 20% of those on Page 1 will move to Page 2.

The evolution is given by $\mathbf{x}_{k+1} = A\mathbf{x}_k$, $k = 1, \ldots$ Suppose we start at $\mathbf{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

What happens as $k \to \infty$?

It turns out that the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 0.5$ and $\lambda_3 = 0.2$ and the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix}.$$

We can write

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3, \qquad \text{where} \quad c_1 = \frac{1}{20}, \ c_2 = \frac{2}{45}, \ c_3 = \frac{1}{36}.$$

Since $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ it follows that $A^k\mathbf{v}_i = \lambda_i^k\mathbf{v}_i$ so

$$A^k\mathbf{x}_0 = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + c_3\lambda_3^k\mathbf{v}_3 = \frac{1}{20}1^k\begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix} + \frac{2}{45}(0.5)^k\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{1}{36}(0.2)^k\begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix} \to \frac{1}{20}\begin{bmatrix} 7 \\ 5 \\ 8 \end{bmatrix},$$

as $k \to \infty$.

A **distribution vector** is a vector with all components positive or 0 and adding up to 1.
A **transition matrix** is a matrix in which each column vector is a distribution vector.

Th Let A be an $n \times n$ transition matrix. Then there is exactly one eigenvector $\mathbf{x}_{equ}$, called the **equilibrium distribution**, with eigenvalue 1. All other eigenvalues are $|\lambda| < 1$.

Let $\mathbf{v}_k$ be the eigenvectors of A, with eigenvalues $\lambda_1 = 1$ and $|\lambda_k| < 1$ for $k > 1$. It follows that the system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ which is initially is in the state $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ will after k times steps be in the state

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

and as $k \to \infty$

$$\mathbf{x}_k \to c_1\mathbf{v}_1.$$