Ex 1 In a certain town, 30% of the married men get divorced each year and 20% of the single men get married each year. Suppose that initially there are 8000 married men and 2000 single men. What is the proportion of married as \( k \to \infty \)?

Sol Let \( w_k = \begin{bmatrix} w_{k1} \\ w_{k2} \end{bmatrix} = \begin{bmatrix} \text{number of married men after } k \text{ years} \\ \text{number of single men after } k \text{ years} \end{bmatrix} \).

Let \( A \) be the \( 2 \times 2 \) matrix such that \( w_{k+1} = Aw_k \).

\[
A = \begin{bmatrix} \text{proportion of married} & \text{proportion of single} \\ \text{proportion of married} & \text{proportion of single} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}
\]

\( w_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} \). After the first year we get \( w_1 = Aw_0 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix} \).

After the second year we get \( w_2 = Aw_1 = A^2w_0 \) and so on:

\[
w_k = A^k w_0
\]

It seems like as \( k \to \infty \), \( w_k \) converges: \( w_{10} = \begin{bmatrix} 4004 \\ 5996 \end{bmatrix}, w_{20} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}, w_{30} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix} \).

In fact, any initial condition will converge to the steady state \((4000, 6000)^T\), for which the number of divorces \(0.3 \cdot 4000\) is equal to the number of marriages \(0.2 \cdot 6000\). If we start with \( x_1 = (2, 3)^T \) proportional to the steady state we get back \( x_1 \):

\[
A x_1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = x_1
\]

There is another vector \( x_2 = (-1, 1)^T \) that \( A \) acts on by simply multiplying by \(1/2\):

\[
A x_2 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2} x_2
\]

The vectors \( x_1, x_2 \) form a basis so we can write our initial condition in terms of these:

\[
w_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = 2000 \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 4000 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2000x_1 - 4000x_2.
\]

Then as \( k \to \infty \)

\[
w_k = A^k w_0 = 2000A^k x_1 - 4000A^k x_2 = 2000x_1 - 4000 \frac{1}{2^k} x_2 \to 2000x_1 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}.
\]

A scalar \( \lambda \) such that \( Ax = \lambda x \) for some \( x \neq 0 \) is called an eigenvalue and a corresponding vector \( x \) is called an eigenvector.

We just calculated \( A^k x \) for large \( k \) using the eigenvalues and eigenvectors.

We express \( x = c_1 x_1 + c_2 x_2 \) in terms of the basis of eigenvectors \( Ax_i = \lambda_i x_i, i = 1, 2 \).

Change of coordinates \( x = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \), where \( P = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \), so \( \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1} x \).

Then \( A^k x = c_1 \lambda_1^k x_1 + c_1 \lambda_2^k x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} ^k \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PD^k P^{-1} x, \) where \( D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \).

Hence \( A = PD P^{-1} \) and \( A^k = (PD P^{-1})^k = PD P^{-1} PD P^{-1} \cdots PD P^{-1} = PD P^{-1} \).
Eigenvectors

A scalar \( \lambda \) such that \( A\mathbf{x} = \lambda \mathbf{x} \) for some \( \mathbf{x} \neq 0 \) is called an **eigenvalue** and a corresponding vector \( \mathbf{x} \) is called an **eigenvector**.

**Ex 2** Let \( L \) be the line in \( \mathbb{R}^2 \) that is spanned by the vector \( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \).

Let \( T \) be the linear transformation that projects any vector orthogonally onto \( L \).

The matrix for \( T \) in the standard coordinate system is \( A = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \).

Find the eigenvectors and eigenvalues.

**Sol** Since the projection leaves the line invariant the vector \( \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \) must be an eigenvector with eigenvalue 1: \( A\mathbf{x}_1 = \mathbf{x}_1 \). Moreover, since the orthogonal vector \( \mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \) is mapped to 0 its also an eigenvector with eigenvalue 0: \( A\mathbf{x}_2 = 0 = 0 \cdot \mathbf{x}_2 \).

If we express \( \mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \), in terms of the basis of eigenvectors then \( A\mathbf{x} = c_1 \mathbf{x}_1 \).

Change of coordinates \( \mathbf{x} = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \), where \( P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \), and \( \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1} \mathbf{x} \),

where \( P^{-1} = \frac{1}{10} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} \).

Hence \( A\mathbf{x} = c_1 \mathbf{x}_1 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PD P^{-1} \), where \( D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).

The matrix \( D \) for \( T \) in the \( B = \{ \mathbf{x}_1, \mathbf{x}_2 \} \) coordinate system is hence very simple.

The matrix for \( A \) for \( T \) in the standard coordinates is more complicated. The following diagram commute

\[ \begin{array}{ccc}
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &=& \begin{bmatrix} \mathbf{x} \end{bmatrix}_B \\
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &=& \begin{bmatrix} \mathbf{x} \end{bmatrix}_B
\end{array} \]

**Ex 3** Let \( T \) be the linear transformation rotating a vector an angle \( \theta \). The matrix for \( T \) is

\[ A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \]. Find the eigenvectors and eigenvalues of \( T \).

**Sol** Unless \( \theta \) is a multiple of \( \pi \) it does not have any real eigenvalues and eigenvectors. If \( \theta \) is a multiply of \( \pi \) the eigenvalues are \( \pm 1 \).
The vectors with eigenvalue 1:

Then as \( k \to \infty \) the vector \( \mathbf{w} \) is called a steady state solution. If \( \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \) have

There is another vector \( \mathbf{x}_2 = (-1,1)^T \) that \( A \) acts on by simply multiplying by \( 1/2 \):

The vectors \( \mathbf{x}_1, \mathbf{x}_2 \) form a basis so we can write our initial condition in terms of these:

Then as \( k \to \infty \)

A scalar \( \lambda \) such that \( A \mathbf{x} = \lambda \mathbf{x} \) for some \( \mathbf{x} \neq 0 \) is called an eigenvalue and a corresponding vector \( \mathbf{x} \) is called an eigenvector.

We just calculated \( A^k \mathbf{x} \) for large \( k \) using the eigenvalues and eigenvectors.

We express \( \mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 \) in terms of the basis of eigenvectors \( A \mathbf{x}_i = \lambda_i \mathbf{x}_i, i=1,2 \).

Change of coordinates \( \mathbf{x} = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \), where \( P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} \), so \( \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1} \mathbf{x} \).

Then \( A^k \mathbf{x} = c_1 \lambda_1^k \mathbf{x}_1 + c_2 \lambda_2^k \mathbf{x}_2 = P \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P D^k P^{-1} \mathbf{x} \), where \( D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \).

Hence \( A = P D P^{-1} \) and \( A^k = (P D P^{-1})^k = P D P^{-1} P D P^{-1} \cdots P D P^{-1} = P D^k P^{-1} \).

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