23. Lecture 23 Diagonalization

Ex 1 In a certain town, 30% of the married men get divorced each year and 20% of the single men get married each year. Suppose that initially there are 8000 married men and 2000 single men. What is the proportion of married as $k \to \infty$?

Sol Let
$$\mathbf{w}_k = \begin{bmatrix} w_{k1} \\ w_{k2} \end{bmatrix} = \begin{bmatrix} \text{number of married men after } k \text{ years} \\ \text{number of single men after } k \text{ years} \end{bmatrix}.$$
Let A be the 2×2 matrix such that
$$\mathbf{w}_{k+1} = A\mathbf{w}_k,$$
$$A = \begin{bmatrix} \text{proportion of married} & \text{proportion of single} \\ \text{that stays married in a year} & \text{that gets married in a year} \\ \text{proportion of married} & \text{proportion of single} \\ \text{that gets divorced in a year} & \text{that stays single in a year} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$
$\mathbf{w}_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}$. After the first year we get $\mathbf{w}_1 = A\mathbf{w}_0 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$.
After the second year we get $\mathbf{w}_2 = A\mathbf{w}_1 = A^2\mathbf{w}_0$ and so on:
$$\mathbf{w}_k = A^k\mathbf{w}_0$$
It seems like as $k \to \infty$, $\mathbf{w}_k$ converges: $\mathbf{w}_{10} = \begin{bmatrix} 4004 \\ 5996 \end{bmatrix}$, $\mathbf{w}_{20} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$, $\mathbf{w}_{30} = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$.
In fact, any initial condition will converge to the **steady state** $(4000, 6000)^T$, for which the number of divorces $0.3 \cdot 4000$ is equal to the number of marriages $0.2 \cdot 6000$. If we start with $\mathbf{x}_1 = (2,3)^T$ proportional to the steady state we get back $\mathbf{x}_1$:
$$A\mathbf{x}_1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{x}_1$$
There is another vector $\mathbf{x}_2 = (-1, 1)^T$ that A acts on by simply multiplying by $1/2$:
$$A\mathbf{x}_2 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}\mathbf{x}_2$$
The vectors $\mathbf{x}_1$, $\mathbf{x}_2$ form a basis so we can write our initial condition in terms of these:
$$\mathbf{w}_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix} = 2000 \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 4000 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2000\mathbf{x}_1 - 4000\mathbf{x}_2.$$
Then as $k \to \infty$
$$\mathbf{w}_k = A^k\mathbf{w}_0 = 2000A^k\mathbf{x}_1 - 4000A^k\mathbf{x}_2 = 2000\mathbf{x}_1 - 4000\frac{1}{2^k}\mathbf{x}_2 \to 2000\mathbf{x}_1 = \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}.$$
A scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$ is called an **eigenvalue** and a corresponding vector $\mathbf{x}$ is called an **eigenvector**.

We just calculated $A^k\mathbf{x}$ for large k using the eigenvalues and eigenvectors.
We express $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ in terms of the basis of eigenvectors $A\mathbf{x}_i = \lambda_i\mathbf{x}_i$, $i = 1, 2$.
Change of coordinates $\mathbf{x} = P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where $P = \begin{bmatrix} | & | \\ \mathbf{x}_1 & \mathbf{x}_2 \\ | & | \end{bmatrix}$, so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}$.
Then $A^k\mathbf{x} = c_1\lambda_1^k\mathbf{x}_1 + c_1\lambda_2^k\mathbf{x}_2 = \begin{bmatrix} | & | \\ \mathbf{x}_1 & \mathbf{x}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PD^kP^{-1}\mathbf{x}$, where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.
Hence $A = PDP^{-1}$ and $A^k = \left(PDP^{-1}\right)^k = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^kP^{-1}$.

Eigenvectors

A scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq 0$ is called an **eigenvalue** and a corresponding vector $\mathbf{x}$ is called an **eigenvector**.

Ex 2 Let L be the line in $\mathbf{R}^2$ that is spanned by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
Let T be the linear transformation that projects any vector orthogonally onto L.
The matrix for T in the standard coordinate system is $A = \frac{1}{10}\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$.
Find the eigenvectors and eigenvalues.
Sol Since the projection leaves the line invariant the vector $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ must be an eigenvector with eigenvalue 1: $A\mathbf{x}_1 = \mathbf{x}_1$. Moreover, since the orthogonal vector $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is mapped to $\mathbf{0}$ its also an eigenvector with eigenvalue 0: $A\mathbf{x}_2 = \mathbf{0} = 0 \cdot \mathbf{x}_2$.

If we express If $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, in terms of the basis of eigenvectors then $A\mathbf{x} = c_1\mathbf{x}_1$.
Change of coordinates $\mathbf{x} = P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where $P = \begin{bmatrix} \mathbf{x}_1 \, \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$, and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\mathbf{x}$, where $P^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$.
Hence $A\mathbf{x} = c_1\mathbf{x}_1 = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PDP^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
The matrix D for T in the $\mathcal{B} = \{\mathbf{x}_1, \mathbf{x}_2\}$ coordinate system is hence very simple.
The matrix for A for T in the standard coordinates is more complicated. The following diagram commute

$$\begin{array}{ccc} c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{x} & \xrightarrow{A} & A\mathbf{x} = c_1\mathbf{x}_1 \\ P\uparrow & & \uparrow P \\ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}} & \xrightarrow{D} & [A\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \end{array},$$

Ex 3 Let T be the linear transformation rotating a vector an angle θ. The matrix for T is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Find the eigenvectors and eigenvalues of T.
Sol Unless θ is a multiple of π it does not have any real eigenvalues and eigenvectors. If θ is a multiply of π the eigenvalues are ± 1.

Summary

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Let A be the 2×2 matrix such that
$$\mathbf{w}_{k+1} = A\mathbf{w}_k, \qquad A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}, \qquad \mathbf{w}_0 = \begin{bmatrix} 8000 \\ 2000 \end{bmatrix}$$
As $k \to \infty$ $\mathbf{w}_k \to \begin{bmatrix} 4000 \\ 6000 \end{bmatrix}$. This is a **steady state solution**. If $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ have
$$A\mathbf{x}_1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \mathbf{x}_1$$
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