22. 6.3 Geometric Interpretation of Determinants

The magnitude of the determinant of a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ is the volume of the n-dimensional parallelepiped with the column vectors as it edges
$P(\mathbf{a}_1, \dots, \mathbf{a}_n) = \{\mathbf{x} \in \mathbf{R}^n; \, \mathbf{x} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n, \, 0 \le c_1 \le 1, \dots, 0 \le c_n \le 1\}$:

$$|\det A| = \text{Vol}\big(P\big)$$

The sign of the determinant depends on the orientation of the column vectors.

In 2 and 3 dimensions it was proven in the multi-variable calculus classes that the magnitude of the cross product of two vectors gives the area and the scalar triple product of three vectors gives the volume. The sign is positive if the vectors form a positively oriented system. The proof here is from Section 3.3 in the text by Lay et al. where there are pictures:

Th If A is a 2×2 matrix, the area of the parallelogram with its columns as its sides is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelogram with its columns as its edges is $|\det A|$.
Pf of the 2×2 cases. The theorem is obviously true for diagonal 2×2 matrices:

$$\left|\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right| = |ad| = \text{Area of rectangle with sides } a \text{ and } d$$

We will show that any 2×2 matrix $A = \begin{bmatrix} \mathbf{a}_1 \, \mathbf{a}_2 \end{bmatrix}$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$. We know that the absolute value of the determinant is unchanged when two columns are interchanged or a multiple of one column is added to another and its easy to see that one can transform A into diagonal form with such operations. Column interchanges do not change the parallelogram at all so it suffices to prove the following fact: The area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$ for any c. This follows from that the points $\mathbf{a}_2$ and $\mathbf{a}_2 + c\mathbf{a}_1$ have the same perpendicular distance to the line through $\mathbf{0}$ and $\mathbf{a}_1$.
The proof in the 3×3 case is similar. It is obviously true in the diagonal case since it is just a cube, and we will argue as before that the volume is unchanged if we add a multiple of one row to another. A parallelepiped is a box with two sloping sides. Its volume is the area of the base in the plane $\text{Span}[\mathbf{a}_1, \mathbf{a}_3]$ times the altitude of $\mathbf{a}_2$ above $\text{Span}[\mathbf{a}_1, \mathbf{a}_3]$. Any vector $\mathbf{a}_2 + c\mathbf{a}_1$ has the same altitude because $\mathbf{a}_2 + c\mathbf{a}_1$ lies in the plane $\mathbf{a}_2 + \text{Span}[\mathbf{a}_1, \mathbf{a}_3]$, which is parallel to $\text{Span}[\mathbf{a}_1, \mathbf{a}_3]$. Hence the volume of the parallelepiped is unchanged when $[\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ is changed to $[\mathbf{a}_1, \mathbf{a}_2 + c\mathbf{a}_1, \mathbf{a}_3]$. Thus a column replacement operation does not affect the volume of the parallelepiped. Since column interchanges have no effect on the volume, the proof is complete.

Limited Gauss-Jordon elimination

In the above we used that to get a matrix in row echelon form and diagonal form if it is invertible it suffices to use two of the row operations:
I) Interchanging two rows changes the sign.
III) Add a multiple of one row to another does not change the determinant.

Determinants of maps

Th Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation $T(\mathbf{x}) = \mathbf{x}$, where A is a 2×2 matrix. If S is a parallelogram in $\mathbf{R}^2$, then

$$\operatorname{Area} T(S) = |\det A| \operatorname{Area} S$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbf{R}^3$, then

$$\operatorname{Vol} T(S) = |\det A| \operatorname{Vol} S$$

By dividing up any area S into smaller squares the theorem holds for any area S.
Pf A parallelogram at the origin in $\mathbf{R}^2$ determined by vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ has the form

$$S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2;\ 0 \le s_1 \le 1,\ 0 \le s_2 \le 1\}$$

The image of S under T consists of points of the form

$$T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) = s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2) = s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2,$$

where $0 \le s_1 \le 1$, $0 \le s_2 \le 1$. It follows that $T(S)$ is the parallelogram determined by the columns of the matrix $[A\mathbf{b}_1\, A\mathbf{b}_2]$. This matrix can be written as AB, where $B = [\mathbf{b}_1, \mathbf{b}_2]$. By the of interpretation of the determinants as area and by the product theorem,

$$\operatorname{Area} T(S) = |\det AB| = |\det A|\,|\det B| = |\det A| \operatorname{Area} S$$

Cramer's rule

Th Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbf{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \det A_i(\mathbf{b})/\det A, \qquad i = 1, \ldots, n,$$

where $A_i(\mathbf{b})$ is the matrix obtained from A by replacing column i by the vector $\mathbf{b}$.

Pf Denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and the columns of the $n \times n$ identity matrix I by $\mathbf{e}_1, \ldots, \mathbf{e}_n$. If $A\mathbf{x} = \mathbf{b}$, the definition of matrix multiplication shows that

$$A\dot{I}_i(\mathbf{x}) = A[\mathbf{e}_1 \ldots \mathbf{x} \ldots \mathbf{e}_n] = [A\mathbf{e}_1 \ldots A\mathbf{x} \ldots A\mathbf{e}_n] = [\mathbf{a}_1 \ldots \mathbf{b} \ldots \mathbf{a}_n] = A_i(\mathbf{b}).$$

By the multiplicative property of determinants,

$$\det A \, \det I_i(\mathbf{x}) = \det A_i(\mathbf{b})$$

The second determinant on the left is simply x_i.

A formula for the inverse using determinants

Cramer's rule leads easily to a general formula for the inverse of an $n \times n$ matrix A. The jth column of A^{-1} is a vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{e}_j$ where $\mathbf{e}_j$ is the jth column of the identity matrix, and the ith entry of $\mathbf{x}$ is the (i, j)th-entry of A^{-1}. By Cramer's rule,

$$(i, j)\text{th-entry of } A^{-1} = \mathbf{x}_i = \det A_i(\mathbf{b})/\det A \tag{22.1}$$

Recall that A_{ji} denotes the submatrix of A formed by deleting row j and column i. A cofactor expansion down column i of $A_i(\mathbf{e}_j)$ shows that

$$\det A_i(\mathbf{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji},$$

where C_{ji} is called a cofactor of A. By (22.1), the (i, j)th-entry of of A^{-1} is the cofactor C_{ji} divided by $\det A$. [Note that the subscripts on C_{ji} are the reverse of (i, j)] Thus

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The matrix of cofactors on the right side above is called the adjugate (or classical adjoint) of A, denoted by $\operatorname{adj} A$.

Cofactor expansion

Let $A = (a_{ij})$ be an $n \times n$ matrix and let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij} and let the **cofactors** be $C_{ij} = (-1)^{i+j} \det(A_{ij})$.

Th For any i and j we have

$$\det A = a_{i1}C_{i1} + \ldots + a_{in}C_{in} = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj}.$$

Pf Since the determinant just changes sign if we switch two rows and the determinant of the transpose is the same it suffices to prove that $\det A = a_{11}C_{11} + \ldots + a_{n1}C_{n1}$. Since the determinant is linear in the rows and hence in the columns it suffices to prove it for the case of $a_{i1} = 1$, for some i but $a_{j1} = 0$ for $j \neq i$. We can however reduce the case $i > 1$ to the case of $i = 1$ by moving the ith row to the top passed each of the $i - 1$ rows above it. Since it requires $i - 1$ switches to it would cause the determinant to changes sign with the factor $(-1)^{i-1}$, which is exactly how the sign would change when going from C_{i1} to C_{11}. It therefore suffices to consider the case when $a_{11} = 1$ but $a_{j1} = 0$ for $j > 1$, which is exactly the case of a block matrix for which we proven that $\det \begin{bmatrix} 1 & * \\ 0 & A_{11} \end{bmatrix} = \det A_{11}$.

Calculating determinants with row reduction and expansion along rows or columns

Ex 1 Find the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

Sol Adding -2 times the 1st row to the 2nd and expanding along the first column:

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -6 & -1 \\ -2 & 0 \end{vmatrix} = 1 \cdot \big((-6)0 - (-1)(-2)\big) = -2$$

Ex 2

$$\begin{vmatrix} 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-2)2 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = (-2)2(1 \cdot 1 - 1 \cdot 0) = -4.$$

Summary

The magnitude of the determinant of a matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ is the volume of the n-dimensional parallelepiped with the column vectors as it edges
$P(\mathbf{a}_1, \dots, \mathbf{a}_n) = \{\mathbf{x} \in \mathbf{R}^n;\ \mathbf{x} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n,\ 0 \le c_1 \le 1, \dots, 0 \le c_n \le 1\}$:

$$\big|\det A\big| = \mathrm{Vol}\,\big(P\big)$$

The sign of the determinant depends on the orientation of the column vectors.
Th If A is a 2×2 matrix, the area of the parallelogram with its columns as its sides is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelogram with its columns as its edges is $|\det A|$. The proof use that to get a matrix in row echelon form, and diagonal form if it is invertible, it suffices to use two of the row operations:
I) Interchanging two rows changes the sign.
III) Add a multiple of one row to another does not change the determinant.
As a consequence of the theorem we have:
Th Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbf{R}^2$, then

$$\mathrm{Area}\, T(S) = |\det A|\, \mathrm{Area}\, S$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbf{R}^3$, then

$$\mathrm{Vol}\, T(S) = |\det A|\, \mathrm{Vol}\, S$$

One typically calculates determinants either with **row operations** or **Cofactor expansion**
Let $A=(a_{ij})$ be an $n\times n$ matrix and let A_{ij} denote the $(n–1)\times(n–1)$ matrix obtained from A by deleting the row and column containing a_{ij}, and let the **cofactors** be $C_{ij} = (-1)^{i+j} \det(A_{ij})$. The **determinant** satisfy

$$\det A = a_{11}C_{11} + \dots + a_{1n}C_{1n}.$$

By switching rows and taking the transpose we get:
Th For any i and j we have

$$\det A = a_{i1}C_{i1} + \dots + a_{in}C_{in} = a_{1j}C_{1j} + \dots + a_{nj}C_{nj}.$$

We have already seen that this is true in the 3 dimensional case. In general it is clear from

$$\det A = \sum\nolimits_\sigma a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}\, \mathrm{sign}\,(\sigma),$$

that the terms are the same so we just need the sign right, see Theorem 6.2.10, Problem 68.

Th (Cramer's rule) Let A be an invertible $n \times n$ matrix. For any $\mathbf{b} \in \mathbf{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \det A_i(\mathbf{b}) / \det A, \qquad i = 1, \dots, n,$$

where $A_i(\mathbf{b})$ is the matrix obtained from A by replacing column i by the vector $\mathbf{b}$. Moreover

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$