# 21. Lecture 21: 6.2 Properties of determinants

We derived a formula for the **determinant** of an $n\times n$ matrix that we now take as definition:

$$\det A = \sum\nolimits_{\sigma} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}\,\mathrm{sign}\,(\sigma),$$

where the sum is over all permutations $\sigma$ of $\{1,2,\dots,n\}$. ($\{\sigma(1),\sigma(2),\dots,\sigma(n)\}$ is a re-orderings of the set $\{1,2,\dots,n\}$. E.g. $\{3,2,5,1,4\}$ is a permutation of $\{1,2,3,4,5\}$.) The sign of a permutation $\sigma$ is defined according to the number of inversions in $\sigma$.
An **inversion** is a pair $\sigma(i) > \sigma(j)$ with $i < j$, i.e. it comes in the wrong order, a bigger one before a smaller one. A permutation is called **even or odd** according two whether the number of **inversions** in its result $\{\sigma(1),...,\sigma(n)\}$ is an even or odd integer. The **sign** of a permutation $\mathrm{sign}\,(\sigma)$ is 1 if the permutation is even and $-1$ if it is odd. (E.g. $\{3,2,5,1,4\}$ have five inversions $(3,2)$, $(3,1)$, $(2,1)$, $(5,1)$ and $(5,4)$ so the sign is $-1$.)

**Th** The determinant of the identity matrix is 1.
**Pf** It is clear that the sign of the identity permutation is 1 since there are no inversion in it.

We now have to check that the so defined sign changes with a simple switch. This is clear if we switch two neighbors, since only the orders of the two changes. The desired result will follow if we show that we can do any switch by an odd number of switches of neighbors. We need $\ell-k$ exchanges of neighbors to move an entry in place $k$ to place $\ell$. Then $\ell-k-1$ exchanges move the one originally in place $\ell$ (and now found in place $\ell-1$) back to place $k$. Since $\ell-k+(\ell-k-1)$ is odd, the result follows. (Check in some simple case to see this.)

**Th** The determinant changes sign if we switch two rows. If two rows are the same it is 0.
**Pf** If we switch say row 1 and 2 then $a_{1\sigma(1)}a_{2\sigma(2)}$ becomes $a_{2\sigma(1)}a_{1\sigma(2)}$ but all other factors remain the same. If we then switch $\sigma(1)$ and $\sigma(2)$ in all the permutations then $a_{2\sigma(1)}a_{1\sigma(2)}$ goes back to $a_{2\sigma(2)}a_{1\sigma(2)}$, all other factors remain the same apart from that the signs of the permutations change by the above discussion. If rows are the same it would be minus itself.

**Th** The determinant is linear in each row when the other rows are fixed, e.g. in the first row:

$$\det\begin{bmatrix} - & \mathbf{x}+\mathbf{y} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = \det\begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} + \det\begin{bmatrix} - & \mathbf{y} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}, \qquad \det\begin{bmatrix} - & k\,\mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = k\det\begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}.$$

**Pf** This follows since each term in the definition is linear in $a_{1\sigma(1)}$ when the others are fixed.

**Th** $\det A^T = \det A$.
**Pf** Using the multi-linearity of the determinant this reduces to proving that $\det P^T = \det P$ for permutation matrices $P$, i.e. matrices with a 1 only at one place in each row, and different places for each row, and 0 everywhere else. Such matrices are in particular orthogonal since the rows are orthonormal so $P^TP = I$. Since the determinant of a permutation matrix is $\pm 1$ depending on the sign of the permutation and since the sign of a composition or product of permutation is the product of the signs it follows that $\det P^T \det P = \det I = 1$. It follows that either $\det P^T = \det P = 1$ or $\det P^T = \det P = -1$.

**Th** If A is a triangular matrix the determinant is the product of the diagonal elements.

Let $A=(a_{ij})$ be an $n\times n$ matrix and let $A_{ij}$ denote the $(n-1)\times(n-1)$ matrix obtained from $A$ by deleting the row and column containing $a_{ij}$, and let $C_{ij} = (-1)^{i+j}\det(A_{ij})$.
**Th** For any $i$ and $j$ we have $\det A = a_{i1}C_{i1} + ... + a_{in}C_{in} = a_{1j}C_{1j} + ... + a_{nj}C_{nj}$.
**Pf** We proved this in the $3\times 3$ case. For the general case we will prove it later.

THEOREM (EFFECT OF ROW OPERATIONS ON DETERMINANTS):
I) Interchanging two rows changes the sign.
II) Multiply a row by a nonzero constant $\alpha$ multiplies the determinant by $\alpha$.
III) Add a multiple of one row to another does not change the determinant.

Only the last statement has not been proven yet but it follows from the linearity

$$\det\begin{bmatrix} - & \mathbf{r}_1+k\mathbf{r}_2 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = \det\begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} + k\det\begin{bmatrix} - & \mathbf{r}_2 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}, \quad \text{where} \quad \det\begin{bmatrix} - & \mathbf{r}_2 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = 0,$$

and (I) since if two rows are the same when you switch them you get minus itself so it is 0.

It will be convenient to express the elementary row operations as multiplication with elementary matrices. Let us illustrate what they are in the $3 \times 3$ case:
Type I: Let $E$ be the elementary matrix that interchanges row 2 and row 3:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Type II: Multiply a row by a nonzero constant $\alpha$ multiplies the determinant by $\alpha$.

$$EA = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Type III: Add a multiple of one row to another does not change the determinant.

$$EA = \begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+\beta a_{31} & a_{12}+\beta a_{32} & a_{13}+\beta a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We have shown (at least in some cases) that if $E$ is an elementary matrix then

$$\det(EA) = \begin{cases} -\det(A), & \text{if } E \text{ is of type I: interchange two rows.} \\ \alpha\det(A), & \text{if } E \text{ is of type II: multiply row by } \alpha. \\ \det(A), & \text{if } E \text{ is of type III: add multiple of row to another.} \end{cases}$$

By letting $E$ act on the identity $I$ in place of $A$, using that $\det(I) = 1$, we get

$$\det(E) = \begin{cases} -1, & \text{if } E \text{ is of type I} \\ \alpha, & \text{if } E \text{ is of type II} \\ 1, & \text{if } E \text{ is of type III} \end{cases}$$

It therefore follows that if $E$ is an elementary matrix

$$\det(EA) = \det(E)\det(A). \tag{21.1}$$

**Th** An $n \times n$ matrix $A$ is invertible if and only if $\det(A) \neq 0$.
**Pf** The matrix $A$ can be reduced to row echelon form $U = E_k \cdots E_1 A$, where $E_i$ are elementary matrices. By repeated use of (21.1) $\det(U) = \det(E_k)\cdots\det(E_1)\det(A)$. Since $\det(E_i) \neq 0$ it follows that $\det(A) \neq 0$ if and only if $\det(U) \neq 0$. If $A$ is not invertible then $U$ has a row consisting entirely of zeros and hence $\det(U) = 0$. If $A$ is invertible, $U$ is nondegenerate triangular with 1's in the diagonal so $\det(U) = 1$.

**Th** $\det(AB) = \det(A)\det(B)$.
**Pf** If $A$ is invertible it can be written as a product of elementary matrices $E_k \cdots E_1$:

$$\det(AB) = \det(E_k \cdots E_1 B) = \det(E_k)\cdots\det(E_1)\det(B) = \det(A)\det(B).$$

**Ex 1** Find the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

**Sol** Adding $-2$ times the 1st row to the 2nd and expanding along the first column:

$$\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 0 \\ 0 & -6 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} -6 & -1 \\ -2 & 0 \end{vmatrix} = 1 \cdot \big((-6)0 - (-1)(-2)\big) = -2$$

**Ex 2** $$\begin{vmatrix} 2 & 0 & 2 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (-2)2 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = (-2)2(1 \cdot 1 - 1 \cdot 0) = -4.$$

## Summary

We derived a formula for the **determinant** of an $n\times n$ matrix that we now take as definition:

$$\det A = \sum_\sigma a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}\,\mathrm{sign}\,(\sigma),$$

where the sum is over all permutations $\{\sigma(1),\sigma(2),\dots,\sigma(n)\}$ of $\{1,2,\dots,n\}$ and the sign is $+1$ or $-1$ depending on if it takes an even or odd number of switches of to get to $\{1,2,\dots,n\}$.

**Th** The determinant of the identity matrix is 1.

**Th** The determinant changes sign if we switch two rows. If two rows are the same it is 0.

**Th** The determinant is linear in each row when the other rows are fixed, e.g. in the first row:

$$\det\begin{bmatrix} - & \mathbf{x}+\mathbf{y} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = \det\begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} + \det\begin{bmatrix} - & \mathbf{y} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}, \qquad \det\begin{bmatrix} - & k\,\mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = k\det\begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}.$$

**Th** If A is a triangular matrix the determinant is the product of the diagonal elements.

**Effect of row operations on determinants:**
I) Interchanging two rows changes the sign.
II) Multiply a row by a nonzero constant $\alpha$ multiplies the determinant by $\alpha$.
III) Add a multiple of one row to another does not change the determinant.

Only the last statement has not been proven yet but it follows from the linearity

$$\det\begin{bmatrix} - & \mathbf{r}_1+k\mathbf{r}_2 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = \det\begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} + k\det\begin{bmatrix} - & \mathbf{r}_2 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}, \quad \text{where} \quad \det\begin{bmatrix} - & \mathbf{r}_2 & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = 0,$$

and (I) since if two rows are the same when you switch them you get minus itself so it is 0.

We will express the elementary row operations as multiplication with elementary matrices:

$$(2)\leftrightarrow(3):\ E_I=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \alpha(1):\ E_{II}=\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (1)+\beta(3):\ E_{III}=\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By the effect of row operations on determinants

$$\det(E_I A) = -\det A, \quad \det(E_{II}A) = \alpha\det A, \quad \det(E_{III}A) = \det A$$

and by applying this to $A=I$:

$$\det E_I = -1, \quad \det E_{II} = \alpha, \quad \det E_{III} = 1$$

Hence in all cases

$$\det(EA) = \det(E)\det(A). \tag{21.2}$$

**Th** An $n\times n$ matrix $A$ is invertible if and only if $\det(A)\neq 0$.

**Pf** The matrix $A$ can be reduced to row echelon form $U = E_k\cdots E_1 A$, where $E_i$ are elementary matrices. By repeated use of (21.2) $\det(U) = \det(E_k)\cdots\det(E_1)\det(A)$. Since $\det(E_i)\neq 0$ it follows that $\det(A)\neq 0$ if and only if $\det(U)\neq 0$. If $A$ is not invertible then $U$ has a row consisting entirely of zeros and hence $\det(U)=0$. If $A$ is invertible, $U$ is nondegenerate triangular with 1's in the diagonal so $\det(U)=1$.

**Th** $\det(AB) = \det(A)\det(B)$ and $\det A^T = \det A$.

**Pf** If $A$ is invertible it can be written as a product of elementary matrices $E_k\cdots E_1$:

$$\det(AB) = \det(E_k\cdots E_1 B) = \det(E_k)\cdots\det(E_1)\det(B) = \det(A)\det(B).$$

We can reduce it to a permutation matrix that satisfy $P^TP=I$ and hence $\det P^T\det P=1$.