## 20. LECTURE 20: 6.1 DETERMINANTS

As it turns out we can calculate a number called determinant for any given $n \times n$ matrix that determines if the matrix is invertible.

$2 \times 2$ **matrices** By a previous theorem $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible if and only if it is row equivalent to the identity $I$. Assume first that $a_{11} \neq 0$.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \sim \quad (2)-(1)a_{21}/a_{11} \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - a_{12}a_{21}/a_{11} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ 0 & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \end{bmatrix}.$$

The resulting matrix is invertible if and only if $a_{11}a_{22} - a_{21}a_{12} \neq 0$. We define

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}. \tag{20.1}$$

Notice that this quantity changes sign but the magnitude stays the same if we switch the two rows. Since switching two rows leads to row equivalent systems it does not effect the invertibility of $A$. The nonvanishing of (1) in case $a_{21} \neq 0$ is therefore also equivalent to that $A$ is invertible. Furthermore if $a_{11} = a_{21} = 0$ then the matrix clearly not invertible and determinant defined above also vanishes.

$3 \times 3$ **matrices** First assume that $a_{11} \neq 0$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \sim \begin{matrix} \\ (2)-(1)\dfrac{a_{21}}{a_{11}} \\ (3)-(1)\dfrac{a_{31}}{a_{11}} \end{matrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ 0 & \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{bmatrix}.$$

A $3 \times 3$ matrix of this form $A = \begin{bmatrix} b & * \\ 0 & B \end{bmatrix}$, where $B$ is a $2 \times 2$ matrix, is invertible if and only if $B$ is invertible and $b \neq 0$, i.e. if $b \det(B) \neq 0$ (In fact to solve the system $A\mathbf{x} = \mathbf{y}$ one first solves the $2 \times 2$ system $B\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_3 \end{bmatrix}$ and substitute $x_2$ and $x_3$ into the first equation.)

It follows that $A$ is invertible if and only if $a_{11} \begin{vmatrix} \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} & \dfrac{a_{11}a_{23} - a_{21}a_{13}}{a_{11}} \\ \dfrac{a_{11}a_{32} - a_{31}a_{12}}{a_{11}} & \dfrac{a_{11}a_{33} - a_{31}a_{13}}{a_{11}} \end{vmatrix} \neq 0$,

i.e. if

$$\frac{1}{a_{11}}\Big(\big(a_{11}a_{22} - a_{21}a_{12}\big)\big(a_{11}a_{33} - a_{31}a_{13}\big) - \big(a_{11}a_{32} - a_{31}a_{12}\big)\big(a_{11}a_{23} - a_{21}a_{13}\big)\Big)$$
$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0.$$

We therefore define

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}. \tag{20.2}$$

Since this only changes sign if we switch two rows and switching two rows does not effect the invertibility we see that the same result is true if any entry in the first column is nonvanishing. But if all vanish the matrix is not invertible and $\det(A) = 0$.

## The cofactor expansion

Notice now that we can write the determinant of a $3\times 3$ matrix in terms of $2\times 2$ ones:

$$\det(A) = a_{11}\big(a_{22}a_{33} - a_{32}a_{23}\big) - a_{12}\big(a_{21}a_{33} - a_{31}a_{23}\big) + a_{13}\big(a_{21}a_{32} - a_{31}a_{22}\big)$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\begin{bmatrix} & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{bmatrix}.$$

This can be interpreted as $a_{11}$ times the determinant of the $2\times 2$ matrix obtained from $A$ by deleting the row and column of $a_{11}$ minus $a_{12}$ times the determinant of the $2\times 2$ matrix obtained by deleting the row and column of $a_{12}$ plus $a_{13}$ times the determinant of the $2\times 2$ matrix obtained by deleting the row and column of $a_{13}$. This is a convenient way to calculate:

**Ex** Find the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.

**Sol**

$$\det(A) = 1\begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5\begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 2 & -4 \\ 0 & -2 \end{vmatrix} = -2 - 0 + 0 = -2.$$

Let $A = (a_{ij})$ be an $n\times n$ matrix and let $A_{ij}$ denote the $(n-1)\times(n-1)$ matrix obtained from $A$ by deleting the row and column containing $a_{ij}$ and let the **cofactors** be $C_{ij} = (-1)^{i+j}\det(A_{ij})$. The **determinant** satisfy

$$\det A = a_{11}C_{11} + \ldots + a_{1n}C_{1n}. \tag{20.3}$$

The formula is recursive; it assumes that we already know the $(n-1)\times(n-1)$ determinants, but we can use the formula repeatedly to reduce to $2\times 2$ determinants that we defined by (20.1) or we can reduce to $1\times 1$ determinants $\det(a) = a$. The signs in $C_{ij}$ are determined by

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Many books take (20.3) as definition of determinant, but we will give a different definition and prove that (20.3) is a consequence. Moreover, one can expand along any row or column:

**Th** For any $i$ and $j$ we have

$$\det A = a_{i1}C_{i1} + \ldots + a_{in}C_{in} = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj}.$$

**Ex** Find the determinant below. **Sol** Expanding along the first column;

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 5 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 0 & 3 & 5 \end{vmatrix} - 0 + 0 - 0 = 1\cdot 2\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 2(2\cdot 5 - 1\cdot 3) = 14.$$

**Th** If $A$ is triangular the determinant is equal to the product of the diagonal elements.

**Ex** $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 1\cdot 2\cdot 2\cdot 5 = 20.$

## Definition with permutations from basic properties

We will now see how we can derive a formula for the determinant from three basic properties:
(1) The determinant of the identity matrix is one, $\det(I)=1$.
(2) The determinant changes sign if we interchange two rows.
As a consequence of (2) it is 0 if two rows are identical since then it would be minus itself.
(3) The determinant is linear in each row when the other rows are fixed, e.g. in the first row:

$$\det\begin{bmatrix} - & \mathbf{x}+\mathbf{y} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = \det\begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} + \det\begin{bmatrix} - & \mathbf{y} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}, \qquad \det\begin{bmatrix} - & k\,\mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix} = k\det\begin{bmatrix} - & \mathbf{x} & - \\ - & \mathbf{r}_2 & - \\ & \vdots & \\ - & \mathbf{r}_n & - \end{bmatrix}.$$

In the 3×3 case these properties follows from (20.2). Using the linearity along the first row:

$$\det(A)=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Using the linearity along the second row one gets for say the first determinant in the right

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

and using linearity along third row on say the second determinant in the right above gives

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{31}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + a_{11}a_{22}a_{32}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{11}a_{22}a_{33}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

where we used linearity and that it vanishes if two rows are the same. Similarly we get

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{11}a_{23}a_{32}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$+ a_{12}a_{23}a_{31}\begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

We have hence reduced calculating the determinant to calculating the determinant for permutation matrices. The permutation matrices can be reduced to the identity matrix by a number of interchanges of rows. The determinant of the identity matrix is one and each interchange of a row just changes the sign of the determinant so the determinants of the permutation matrices are either 1 or $-1$ depending on if it takes and even or odd number of changes of rows to get to the identity matrix. We therefore define the sign of a permutation to be $\pm 1$ depending on if it takes an even or odd number of row exchanges to get to the identity. This procedure in general gives us a formula for the determinant of an $n\times n$ matrix;

$$\det A = \sum_\sigma a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}\,\mathrm{sign}\,(\sigma),$$

where the sum is over all permutations $\sigma$ of $\{1,2,\ldots,n\}$. It is apriori not clear that the sign of a permutation is well defined, since it is not obvious that you can't turn a permutation into the identity by both an even and an odd number of row exchanges. We will prove this.

## The sign of a permutation

The sign of a permutation is not yet well defined, since we have not proven that you can't switch $\sigma$ to its natural order by both an even and an odd number of switches.
Let us therefore define the sign of a permutation in such a way that its clear that it satisfies that it changes sign under a switch and the sign of the identity permutation is one.

The sign of a permutation $\sigma$ is defined according to the number of inversions in $\sigma$.
An **inversion** is a pair $\sigma(i) > \sigma(j)$ with $i < j$, i.e. it comes in the wrong order, a bigger one before a smaller one. A permutation is called **even or odd** according two whether the number of **inversions** in its result $\{\sigma(1), ..., \sigma(n)\}$ is an even or odd integer. The **sign** of a permutation $\operatorname{sign}(\sigma)$ is 1 if the permutation is even and $-1$ if it is odd. (E.g. $\{3,2,5,1,4\}$ have five inversions $(3,2)$, $(3,1)$, $(2,1)$, $(5,1)$ and $(5,4)$ so the sign is $-1$.)

It is clear that the sign of the identity permutation is 1 since there are no inversion in it.
We just have to check that the so defined sign changes with a simple switch. This is clear if we switch two neighbors, since only the orders of the two changes. The desired result will follow if we show that we can do any switch by an odd number of switches of neighbors. We need $\ell - k$ exchanges of neighbors to move an entry in place $k$ to place $\ell$. Then $\ell - k - 1$ exchanges move the one originally in place $\ell$ (and now found in place $\ell - 1$) back to place $k$. Since $\ell - k + (\ell - k - 1)$ is odd, the result follows. (Check in some simple case to see this.)

## Determinants of block matrices

**Th** Suppose that $A$ are $D$ square matrices then $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \det A \det D$.
**Pf** Since $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & B \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ the proof follows from the fact, proven later, that the determinant of a product of matrices is the product of the determinants, and the lemma:

**Lem** If $A$ is a square matrix and $I$ the identity set $M = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ or $M = \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix}$ then $\det M = \det A$.
**Pf** We will prove the case $M = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$ as the other case is just a relabelling of the coordinates.
We will prove in case $I$ is $1\times 1$, and the case $k\times k$, for $k>1$ follows from repeated use of this.
Let $A = (a_{ij})$ be an $n\times n$ matrix, then $M = (m_{ij})$ is the $(n+1)\times(n+1)$ matrix such that $m_{ij} = a_{ij}$ and $m_{(n+1)j} = m_{i(n+1)} = 0$, for $1 \le i, j \le n$ and $m_{(n+1)(n+1)} = 1$. From the definition:

$$\det M = \sum\nolimits_{\sigma} m_{1\sigma(1)} \cdots m_{n\sigma(n)} m_{(n+1)\sigma(n+1)} \operatorname{sign}(\sigma),$$

where the sum is over all permutations $\{\sigma(1), \ldots, \sigma(n), \sigma(n+1)\}$ of $\{1, \ldots, n, n+1\}$, and the sign of $\sigma$ is defined according to the number of inversions in $\sigma$.
Since $m_{(n+1)\sigma(n+1)} \neq 0$ only when $\sigma(n+1) = n+1$ the sum is only over such permutations.
Moreover, the sign of the permutation $\{\sigma(1), \ldots, \sigma(n), \sigma(n+1)\}$ of $\{1, \ldots, n, n+1\}$, with $\sigma(n+1) = n+1$, is the same as the sign of the permutation $\{\sigma(1), \ldots, \sigma(n)\}$ of $\{1, \ldots, n\}$, because $\sigma(n+1)$ does not determine any inversions since $\sigma(j) < \sigma(n+1) = n+1$, for $j < n+1$.
Since $m_{(n+1)(n+1)} = 1$ we conclude that

$$\det A = \sum\nolimits_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \operatorname{sign}(\sigma),$$

where the sum is now over all permutations $\{\sigma(1), \ldots, \sigma(n)\}$ of $\{1, \ldots, n\}$.

## Summary

In $A$ is an $n \times n$ matrix we define a number $\det(A)$, called the determinant of $A$, such that $\det(A) \neq 0$ is equivalent to that $A$ is invertible.

The determinant $\det(A)$ of a $2 \times 2$ matrix $A$ is $\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$.

The determinant $\det(A)$ of a $3 \times 3$ matrix $A$ is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Note that the $a_{1j}$ is multiplied by the determinant of the $2 \times 2$ matrix obtained by deleting the row and column that $a_{1j}$ is in. In general we give a recursive definition:

Let $A=(a_{ij})$ be an $n{\times}n$ matrix and let $A_{ij}$ denote the $(n–1){\times}(n–1)$ matrix obtained from $A$ by deleting the row and column containing $a_{ij}$, and let the **cofactors** be $C_{ij} = (-1)^{i+j}\det(A_{ij})$. The **determinant** saisfy

$$\det A = a_{11}C_{11} + ... + a_{1n}C_{1n}.$$

For any $i$ and $j$ we have

$$\det A = a_{i1}C_{i1} + ... + a_{in}C_{in} = a_{1j}C_{1j} + ... + a_{nj}C_{nj}.$$

One can derive a formula for the determinant from three basic properties:
(1) The determinant of the identity matrix is one, $\det(I) = 1$.
(2) The determinant changes sign if we interchange two rows.
(3) The determinant is linear in each row when the other rows are fixed.

$2 \times 2$ case: Using the linearity first along the first row and then along the second gives

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 \\ 0 & a_{22} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ a_{21} & 0 \end{vmatrix} + \begin{vmatrix} 0 & a_{12} \\ 0 & a_{22} \end{vmatrix}$$

$$= a_{11}a_{21} \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + a_{11}a_{22} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_{12}a_{21} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + a_{12}a_{22} \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + a_{12}a_{21} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

since by property (2) the determinant vanish if two rows are the same. Similarly we get

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$+ a_{12}a_{23}a_{31} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}.$$

We have hence reduced calculating the determinant to calculating the determinant for permutation matrices. The permutation matrices can be reduced to the identity matrix by a number of interchanges of rows. The determinant of the identity matrix is 1 and each interchange of rows just changes the sign of the determinant so the determinants of the permutation matrices are either 1 or $-1$ depending on if it takes and even or odd number of changes of rows to get to the identity matrix. We therefore define the sign of a permutation to be $\pm 1$ depending on if it takes an even or odd number of row exchanges to get to the identity. We have obtained the following formula for the determinant of an $n \times n$ matrix;

$$\det A = \sum_{\sigma} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)} \operatorname{sign}(\sigma),$$

where the sum is over all permutations $\sigma$ of $\{1, 2, \dots, n\}$. If we make this our definition we have to make sure the sign is well defined: We define the sign of a permutation $\sigma$ according to the number of inversions in $\sigma$. An **inversion** is a pair $\sigma(i) > \sigma(j)$ with $i < j$. With this definition it is easy to see that the sign changes with a switch of two rows, see discussion.