10. Lecture 10: 3.3 Basis and Dimension

Recall that $\mathbf{b}_1,...,\mathbf{b}_n$ form a **basis** for a subspace V if they span V and are linearly independent.

Th If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a spanning set for V then any collection of p vectors $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$, where $p > n$, are linearly dependent.
Pf Since $\mathbf{v}_1, ..., \mathbf{v}_n$ span V we can write each $\mathbf{u}_i$ as a linear combination:

$$\mathbf{u}_i = a_{i1}\mathbf{v}_1 + ... + a_{in}\mathbf{v}_n$$

A linear combination of the $\mathbf{u}_i$ can be written

$$\begin{aligned} c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p &= c_1\big(a_{11}\mathbf{v}_1 + \cdots + a_{1n}\mathbf{v}_n\big) + \cdots + c_p\big(a_{p1}\mathbf{v}_1 + \cdots + a_{pn}\mathbf{v}_n\big) \\ &= \big(c_1a_{11} + \cdots + c_pa_{p1}\big)\mathbf{v}_1 + \cdots + \big(c_1a_{p1} + \cdots + c_pa_{pn}\big)\mathbf{v}_n \end{aligned}$$

Hence $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$ if

$$\begin{aligned} c_1a_{11} + \cdots + c_pa_{p1} &= 0 \\ &\vdots \\ c_1a_{1n} + \cdots + c_pa_{pn} &= 0 \end{aligned}$$

Since $p > n$ this is a homogenous system for $c_1, \cdots, c_p$ with more unknowns than equations so it has a nontrivial solution. Hence there are constants $c_1, ..., c_p$ not all zero such that $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$, i.e. $\mathbf{u}_1, ..., \mathbf{u}_p$ are linearly dependent.

Cor If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ are bases for V then $n = m$.

The number of elements in a basis for V is called the **dimension** of V, written $\dim V$.

Ex Subspaces of $\mathbf{R}^3$:
1-**dimension** Span$\{\mathbf{v}\}$ is a line through the origin.
2-**dimensions** Span$\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u}, \mathbf{v}$ are not parallel is a plane through the origin.
3-**dimensions** Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, where $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ are linearly independent is all of $\mathbf{R}^3$.

The Basis Theorem Let V be an n dimensional subspace. Any set of n vectors that spans V is a basis. Any linearly independent set of n vectors in V is a basis.

Ex 1 Find a basis for the column space of the matrix $B = \begin{bmatrix} | & | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Sol This is easy since matrix is in reduced row echelon form. One sees immediately that $\mathbf{b}_2 = 2\mathbf{b}_1$ and $\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3$, i.e $\mathbf{b}_2$ and $\mathbf{b}_4$ are redundant. Moreover $\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5$ are linearly independent, because each of them has a 1 in a component where the other two are 0 so it is impossible to write any of them as a linear combination of the other two. Alternatively

$$x_1\mathbf{b}_1 + x_3\mathbf{b}_3 + x_5\mathbf{b}_5 = x_1\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + x_3\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + x_5\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = \begin{bmatrix}x_1\\x_3\\x_5\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad \Rightarrow \quad x_1 = x_3 = x_5 = 0,$$

so they are linearly independent. It follows that $\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5$ form a basis for the column space.

Basis for column space

Ex 2 Describe the column space of

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & -1 & 3 & 0 \\ 3 & 6 & 2 & 22 & 1 \\ 4 & 8 & 0 & 16 & 0 \end{bmatrix}$$

Sol Are the columns of A linearly independent?
The columns are linearly dependent if $A\mathbf{x} = 0$ has a nontrivial solution.
Row reduction on the augmented matrix gives that x_2, x_4 are free variables;

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 & 0 \\ 2 & 4 & -1 & 3 & 0 & 0 \\ 3 & 6 & 2 & 22 & 1 & 0 \\ 4 & 8 & 0 & 16 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & -5 & 0 & 0 \\ 0 & 0 & 2 & 10 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 \quad + \quad 4x_4 = 0 \\ \qquad\qquad x_3 + 5x_4 = 0 \\ \qquad\qquad\qquad x_5 = 0 \end{cases} \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ -5x_4 \\ x_4 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} = x_2\mathbf{w}_1 + x_4\mathbf{w}_2$$

i.e. $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions so the columns are linearly dependent.
If $x_2 = 1$, $x_4 = 0$ then $A\mathbf{x} = -2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$, if $x_2 = 0$, $x_4 = 1$ then $A\mathbf{x} = -4\mathbf{a}_1 - 5\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$
Quest Can we find a linearly independent subset of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ that span Col A?
Using the **linear dependency relations** $-2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $-4\mathbf{a}_1 - 5\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$ we see that $\mathbf{a}_2 = 2\mathbf{a}_1$ and $\mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$. If $\mathbf{y} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ then $\mathbf{y} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4 + c_5\mathbf{a}_5 = (c_1 + 2c_2 + 4c_4)\mathbf{a}_1 + (c_3 + 5c_4)\mathbf{a}_3 + c_5\mathbf{a}_5$. Hence $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$.
Are $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ linearly independent? To answer this we use the reduced row echelon form:

$$A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & -1 & 3 & 0 \\ 3 & 6 & 2 & 22 & 1 \\ 4 & 8 & 0 & 16 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ | & | & | & | & | \end{bmatrix} = B$$

Elementary row operations do not affect linear dependency relations among the columns: $A\mathbf{x} = \mathbf{0} \Leftrightarrow B\mathbf{x} = \mathbf{0}$, i.e. $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 + x_5\mathbf{a}_5 = \mathbf{0} \Leftrightarrow x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_3\mathbf{b}_3 + x_4\mathbf{b}_4 + x_5\mathbf{b}_5 = \mathbf{0}$.
$\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5$ are linearly independent since the only solution to $x_1\mathbf{b}_1 + x_3\mathbf{b}_3 + x_5\mathbf{b}_5 = \mathbf{0}$ is $x_1 = x_3 = x_5 = 0$ (since each have a 1 in a component where the others are 0, Ex 1). Hence the only solution to $x_1\mathbf{a}_1 + x_3\mathbf{a}_3 + x_5\mathbf{a}_5 = \mathbf{0}$ is $x_1 = x_3 = x_5 = 0$, so $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ are linearly independent.

Th The pivot columns of A form a basis for Col A.

To form a basis for Nul$A$ use row operations to find the reduced row echelon form $[A\,\mathbf{0}] \sim [B\,\mathbf{0}]$ and use it to find the general solution of $A\mathbf{x} = \mathbf{0}$. The vectors in its parametric form is a basis.
To find a basis for Col A, use row operations to find the reduced row echelon form $A \sim B$ and use it to find the pivot columns. The columns of A corresponding to these form a basis.
$\mathrm{rank}\, A = \dim \mathrm{Col}\, A =$ number of pivot columns of A.
$\mathrm{nullity}\, A = \dim \mathrm{Nul}\, A =$ number of nonpivot columns of A.
Since the number of pivot columns plus the number of nonpivot columns is equal to the total number of columns we have proven:

The Rank Theorem If A is an $m \times n$ matrix then $\mathrm{rank}\, A + \mathrm{nullity}\, A = n$.

The row space (Optional for now)

Let us instead write A and its row reduced form B in terms of the rows

$$A=\begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ - & \mathbf{r}_3 & - \\ - & \mathbf{r}_4 & - \end{bmatrix}=\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & -1 & 3 & 0 \\ 3 & 6 & 2 & 22 & 1 \\ 4 & 8 & 0 & 16 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} - & \mathbf{s}_1 & - \\ - & \mathbf{s}_2 & - \\ - & \mathbf{s}_3 & - \\ - & \mathbf{s}_4 & - \end{bmatrix}=B$$

The **row space** of an $m\times n$ matrix A is the subspace of $\mathbf{R}^n$ spanned by the row vectors of A. The row space of the row reduced form of a matrix is the same as the row space of the matrix since the row vectors of the reduced matrix are formed by taking linear combinations of the row vectors in such a way that the process is reversible.

In this case $\mathrm{Span}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)=\mathrm{Span}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3,\mathbf{s}_4)=\mathrm{Span}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3)$. Further it is clear that $\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3$ are linearly independent (this is because each of the vectors have a 1 in a component where the others are 0). Therefore $\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3$ form a basis for the row space of A.

For an $m\times n$ matrix A the row space $\mathrm{Row}\,A$ and the null space $\mathrm{Nul}\,A$ are both subspaces of $\mathbf{R}^n$, that coincide with the row respectively null space of its reduced row echelon form B. Moreover these spaces are the orthogonal complement of each other. This is seen from the row picture of matrix multiplication. The null space consists of all the $\mathbf{x}$ that satisfy $A\mathbf{x}=\mathbf{0}$:

$$A\mathbf{x}=\begin{bmatrix} - & \mathbf{r}_1 & - \\ - & \mathbf{r}_2 & - \\ - & \mathbf{r}_3 & - \\ - & \mathbf{r}_4 & - \end{bmatrix}\begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix}=\begin{bmatrix} \mathbf{r}_1\cdot\mathbf{x} \\ \mathbf{r}_2\cdot\mathbf{x} \\ \mathbf{r}_3\cdot\mathbf{x} \\ \mathbf{r}_4\cdot\mathbf{x} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e. all $\mathbf{x}$ that are orthogonal to all the row vectors $\mathbf{r}_i\cdot\mathbf{x}=0$ for $i=1,2,3,4$ and hence $\mathbf{r}\cdot\mathbf{x}=0$ for all $\mathbf{r}\in\mathrm{Row}\,A$ and $\mathbf{x}\in\mathrm{Nul}\,A$.

The vectors $\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3$ considered as column vectors in $\mathbf{R}^5$ form a basis for the row space of A. The vectors $\mathbf{w}_1,\mathbf{w}_2$ in Ex 2 form a basis for the null space of A. We claim that $\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3,\mathbf{w}_1,\mathbf{w}_2$ form a basis for $V=\mathrm{Span}(\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3,\mathbf{w}_1,\mathbf{w}_2)$ so $\dim V=5$ and hence $V=\mathbf{R}^5$. First, it follows from the fact the row space and the null space are orthogonal, that there is only one way to write a vector in $\mathbf{x}\in V$ as $\mathbf{x}=\mathbf{r}+\mathbf{w}$, where $\mathbf{r}\in\mathrm{Row}A$ and $\mathbf{w}\in\mathrm{Nul}\,A$. In fact, if there was two ways $\mathbf{r}+\mathbf{w}=\mathbf{r}'+\mathbf{w}'$ then $\mathbf{r}-\mathbf{r}'=\mathbf{w}'-\mathbf{w}$ would be both in the row space and in the null space, but this is impossible since the only vector that is orthogonal to itself is $\mathbf{0}$. Then $\mathbf{r}$ can be expressed uniquely as a linear combination of $\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3$ and $\mathbf{w}$ can be expressed uniquely as a linear combination of $\mathbf{w}_1,\mathbf{w}_2$. It follows that every vector $\mathbf{x}\in V$ can be expressed uniquely as a linear combination of $\mathbf{s}_1,\mathbf{s}_2,\mathbf{s}_3,\mathbf{w}_1,\mathbf{w}_2$ and they therefore form a basis for V.

The transpose A^T of a matrix A is the matrix with rows and columns interchanged, $(A^T)_{ij}=(A)_{ji}$. Hence the column space of A becomes the row space of A^T and therefore we can alternatively find a basis for the column space of A by performing row operations on A^T.

Summary and Questions

Recall that $\mathbf{b}_1,...,\mathbf{b}_n$ form a **basis** for a subspace V if they span V and are linearly independent.

Th If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a spanning set for V then any collection of p vectors $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$, where $p > n$, is linearly dependent.
The proof of this theorem just boils down to that a homogeneous system with more unknowns than equations has a nontrivial solution.
Cor If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ are bases for V then $n = m$.

The number of elements in a basis for V is called the **dimension** of V, written $\dim V$.

The Basis Theorem Let V be an n dimensional subspace. Any set of n vectors that spans V is a basis. Any linearly independent set of n vectors in V is a basis.

How do find a basis for the column space of a matrix A?
Perform row operations $A = \begin{bmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & -1 & 3 & 0 \\ 3 & 6 & 2 & 22 & 1 \\ 4 & 8 & 0 & 16 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} | & | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ | & | & | & | & | \end{bmatrix} = B$
Since $\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5$ are a basis for the column space of B it follows that $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ form a basis for the column space of A. In fact $A\mathbf{x} = \mathbf{0}$ has the same solution set as $B\mathbf{x} = \mathbf{0}$.
Since $\mathbf{b}_2 = 2\mathbf{b}_1$ and $\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_1$ it follows that $\mathbf{a}_2 = 2\mathbf{a}_1$ and $\mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_1$ are redundant. Moreover, $\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5$ are linearly independent since the only solution to $x_1\mathbf{b}_1 + x_3\mathbf{b}_3 + x_5\mathbf{b}_5 = \mathbf{0}$ is $x_1 = x_3 = x_5 = 0$, and therefore the only solution to $x_1\mathbf{a}_1 + x_3\mathbf{a}_3 + x_5\mathbf{a}_5 = \mathbf{0}$ is $x_1 = x_3 = x_5 = 0$, so $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ are linearly independent. We conclude that $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ form a basis for the column space.

It follows that the dimension of the column space the **rank** is the number of lead variables.

The dimension of the null space, called the **nullity** is the number of free variables.

The Rank Theorem If A is an $m \times n$ matrix then $\operatorname{rank} A + \operatorname{nullity} A = n$.