Solutions to Math 201 Final Spring 09

1. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.
(b) Is A diagonalizable? explain why or why not?
Sol. (a) Subtracting the third row from the second and expanding along the second gives:

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & -\lambda & \lambda \\ 1 & 0 & 1-\lambda \end{vmatrix} = (-1)^{2+2}(-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} + (-1)^{2+3}\lambda\begin{vmatrix} 1-\lambda & 2 \\ 1 & 0 \end{vmatrix}$$

$$= -\lambda\big((1-\lambda)^2 + 1 - 2\big) = -\lambda^2(\lambda - 2),$$

so the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$.
(b) For A to be diagonalizable the dimension of the Eigenspace corresponding to the double eigenvalue 0 has to be 2. We hence wish to solve $A\mathbf{x} = \mathbf{0}$. Row reduction gives the system $x_1 + x_3 = 0$ and $x_2 - x_3 = 0$, which can't be reduced further. Since the only free variable is x_3 the eigenspace is one dimensional. Therefore the matrix can not be diagonalized.

2. Let A be a 2×2 matrix with eigenvalues $1/2$ and $-1/2$.
Let Ker $\left(A - \frac{1}{2}I\right) = \text{Span}\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$ and Ker $\left(A + \frac{1}{2}I\right) = \text{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.

(a) Let $\mathbf{x}(t+1) = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = A\mathbf{x}(t)$. Given that $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $\mathbf{x}(3)$.
(b) Draw the phase portrait for the discrete system in part (a).
Sol. (a) We have $\mathbf{x}(k) = A^k\mathbf{x}(0)$. We want to write $\mathbf{x}(0) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the eigenvectors corresponding to the eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/2$ respectively. Then $\mathbf{x}(k) = A^k\mathbf{x}(0) = c_1A^k\mathbf{b}_1 + c_2A^k\mathbf{b}_2 = c_1\lambda_1^k\mathbf{b}_1 + c_2\lambda_2^k\mathbf{b}_2$.
Solving the system $\mathbf{x}(0) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ for c_1 and c_2 gives $c_1 = 0$ and $c_2 = 1$ so $\mathbf{x}(3) = (-1/2)^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

(a) Given that $\lambda = 1$ and 5 are the only eigenvalues of A. Find an orthonormal basis of $\mathbb{R}^3$ denoted by $\mathcal{B}$ consisting of eigenvectors of A.
(b) Given the following quadratic form $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 3x_2^2$.
Describe q in terms of $\mathcal{B}$ coordinates. Show work.
Sol.(a) Ker $(A - I) = \text{Span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ and Ker $(A - 5I) = \text{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$, so $\mathbf{b}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are orthonormal.
(b) We have $A = QDQ^T$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix}$. Set $\mathbf{y} = Q^T\mathbf{x}$.
Then $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, QDQ^T\mathbf{x}\rangle = \langle Q^T\mathbf{x}, DQ^T\mathbf{x}\rangle = \langle \mathbf{y}, D\mathbf{y}\rangle = y_1^2 + 5y_2^2 = \widetilde{q}(\mathbf{y})$.

4. Let f denote a infinitely differentiable function on $\mathbb{R}$. Find all real solutions to the following differential equation $\dfrac{d^2f}{dt^2} - f(t) = 0$.
Sol We have not covered this so we don't give a solution.

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix}$.

(a) Find the inverse of A, if it exists.
(b) Give a basis of the Image of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(x) = Ax$.
Sol. (a) We have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 1 & -3 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ (2)-2(1) \\ (3)-3(1) \end{array} \Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} \\ \\ (3)-(2) \end{array}$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & -6 & -3 & 4 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} (1)-(3) \\ (2)+4(3) \\ \end{array} \quad \Rightarrow \quad A^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -6 & -3 & 4 \\ -1 & -1 & 1 \end{bmatrix}$$

Rem. We used that the same row operations that turn A into the identity I also turn the identity I into A^{-1}. This is because row operations correspond to multiplying with elementary matrices so if $E_3E_2E_1A = I$ and $B = E_3E_2E_1I = E_3E_2E_1$ then $BA = I$, i.e. $B = A^{-1}$.

(b) Since A is invertible T is invertible and the image of T is all of $\mathbb{R}^3$ so the standard basis of $\mathbb{R}^3$ is a basis of the image.

6. Consider $\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 1 & 1 & h \end{array}\right]$

(a) Given that the above is the augmented matrix of a system of equations, find h such that it is consistent.
(b) For $h = 0$ find the least squares solution to the system.

Sol. Row reduction gives $\left[\begin{array}{cc|c} 1 & 1 & -2 \\ 1 & 2 & 1 \\ 1 & 1 & h \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 1 & 1 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & h+2 \end{array}\right]$

For this to be consistent we must have $h = -2$.
(b) The least square 'solution' to $A\mathbf{x} = \mathbf{b}$ is the solution to the normal equation $A^TA\mathbf{x}^* = A^T\mathbf{b}$.
In this case $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ so $A^TA = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and the normal equation is $\begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. We have $\begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix}$ so $\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & -4 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Remark The least square 'solution' $\mathbf{x}^*$ to $A\mathbf{x} = \mathbf{b}$ is not a solution $A\mathbf{x} = \mathbf{b}$ but instead it is the $\mathbf{x}$ that makes $\|A\mathbf{x} - \mathbf{b}\|^2$ as small as possible, i.e. $\|A\mathbf{x} - \mathbf{b}\|^2 \geq \|A\mathbf{x}^* - \mathbf{b}\|^2$, for all $\mathbf{x}$.
Since $\mathbf{p} = \mathrm{Proj}_{\mathrm{Im}\, A}\, \mathbf{b}$ is the closet point to $\mathbf{b}$ in the image of A it follows that $A\mathbf{x}^* = \mathbf{p}$.
Since $\mathbf{p}$ is the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Im}\, A$ it follows that $\mathbf{p} - \mathbf{b}$ is orthogonal to $\mathrm{Im}\, A$ from which it follows that $A^T(\mathbf{p} - \mathbf{b}) = \mathbf{0}$. (In fact $\langle A^T(\mathbf{p} - \mathbf{b}), \mathbf{y} \rangle = \langle \mathbf{p} - \mathbf{b}, A\mathbf{y} \rangle = 0$ for all $\mathbf{y}$.)
It follows that $A^TA\mathbf{x}^* = A^T\mathbf{b}$.

7. Let $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\-2\end{bmatrix}\right\}$ be two different bases of a subspace W in $\mathbb{R}^3$.

(a) Which of the two sets are orthogonal? Show work.

(b) Let $\mathbf{y} = \begin{bmatrix}1\\2\\0\end{bmatrix}$. Is $\mathbf{y} \in W$?

(c) Find $\text{proj}_W\, \mathbf{y}$, that is, the orthogonal projection of $\mathbf{y}$ onto W.

Sol. (a) Let $\mathbf{b}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Then $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$ so they are orthogonal.

(b) That $\mathbf{y} \in W$ is equivalent to that there are comstants c_1, c_2 such that $\mathbf{y} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ i.e. $c_1\begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix}$. The second component says that $c_2 = 2$ and the third component says that $c_1 = c_2 = 2$ but the first components says that $c_1 + c_2 = 2 + 2 = 1$ which is not possible.

(c) The vectors $\mathbf{u}_1 = \mathbf{b}_1/\|\mathbf{b}_1\|$ and $\mathbf{u}_2 = \mathbf{b}_2/\|\mathbf{b}_2\|$ form an orthonormal set. The projection is

$$\text{proj}_W\, \mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2 = \frac{\mathbf{y}\cdot\mathbf{b}_1}{\|\mathbf{b}_1\|^2}\mathbf{b}_1 + \frac{\mathbf{y}\cdot\mathbf{b}_2}{\|\mathbf{b}_2\|^2}\mathbf{b}_2 = \frac{1}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix} + \frac{3}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3/2\\1\\1/2\end{bmatrix}.$$

8. Let A be a 2×2 matrix with eigenvalues 1 and 3, such that $\text{Ker}\,\big(A - I\big) = \text{Span}\{\begin{bmatrix}-1\\1\end{bmatrix}\}$ and $\text{Ker}\,\big(A - 3I\big) = \text{Span}\{\begin{bmatrix}1\\-3\end{bmatrix}\}$.

(a) Find A. Show work.

(b) Let T denote the transformation $T\mathbf{x} = A\mathbf{x}$. Write down the matrix of the transformation T with respect to the basis $\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\-3\end{bmatrix}\}$. Show work.

Sol. (a) Since the eigenvalues are different A can be diagonalized $A = SDS^{-1}$, where $D = \begin{bmatrix}1&0\\0&3\end{bmatrix}$ and $S = \begin{bmatrix}-1&1\\1&-3\end{bmatrix}$ and hence $S^{-1} = \frac{1}{2}\begin{bmatrix}-3&-1\\-1&-1\end{bmatrix}$. Hence $A = \dots$.

(b) The matrix is the matrix D in (a).

9. Answer the following in short. Give justification for your answers.
(i) Let $\mathcal{D}$ denote the space of differentiable functions from $\mathbb{R} \to \mathbb{R}$. Is the function $\langle\ ,\ \rangle : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ defined as $\langle f, g\rangle = f(0)g'(0) + f'(0)g(0)$ an inner product on $\mathcal{D}$?
(ii) Let $V = \text{Span}\left\{\begin{bmatrix}-1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\3\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\-2\end{bmatrix}\right\}$. Find the dimension of V. Explain your answer.
(iii) Let A be a 2×2 matrix with eigenvalues $-1\pm2i$. Then consider the system of differential equations,

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = A\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

What happens to $x(t)$ as $t \to \infty$? Show work.
(iv) Let A be a 2×2 matrix such that $A^3 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$. Then find $\text{Ker}\, A$.
Sol. (i) No because we can have $\langle f, f\rangle = 0$ with $f \neq 0$.
(ii) The second and third vector span the plane $x_1 = 0$ since none is a multiple of the other. However the first vector does not lie in this plane since the first component is not 0, therefore the three vectors span 3 dimensions so they are linearly independent.
(iii) We didn't do systems of differential equations so we don't give the solution to this part.
(iv) Since $(\det A)^3 = \det A^3 = \det I = 1$ it follows that $\det A \neq 0$ so A is invertible.

10. State true or false with justification.
(i) If A is a orthogonal 3×3 matrix then $\det A > 0$.
(ii) Let $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ and $\mathbf{w}_2 \in \text{Span}\{\mathbf{w}_1, \mathbf{w}_3\}$ then $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_3\}$.
(iii) Let $T : V \to W$ be an invertible linear transformation from a vector space V to another vector space W. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent subset of V , then $\{T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3\}$ is a linearly independent set in W.
(iv) If A is a 2×2 symmetric matrix then all its eigenvalues are positive real numbers.
Sol. (i) False, since Q could be a reflection, even just $-I$.
(ii) True, since if $\mathbf{v} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3$ and $\mathbf{w}_2 = d_1\mathbf{w}_1 + d_3\mathbf{w}_3$
then $\mathbf{v} = (c_1 + c_2d_1)\mathbf{w}_1 + (c_3 + d_3c_2)\mathbf{w}_3$.
(iii) True, since T is invertible $c_1T\mathbf{v}_1 + c_2T\mathbf{v}_2 + c_3T\mathbf{v}_3 = T\big(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3\big) = \mathbf{0}$ implies $c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$, which since they are linearly independent implies that $c_1=c_2=c_3=0$.
(iv) False, since again we could take $A = -I$.