Solutions to Questions for Midterm III Review from Final Spring 09

1. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.
(b) Is A diagonalizable? explain why or why not?
Sol. (a) Subtracting the third row from the second and expanding along the second gives:

$$\begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & -\lambda & \lambda \\ 1 & 0 & 1-\lambda \end{vmatrix} = (-1)^{2+2}(-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} + (-1)^{2+3}\lambda\begin{vmatrix} 1-\lambda & 2 \\ 1 & 0 \end{vmatrix}$$
$$= -\lambda\big((1-\lambda)^2 + 1 - 2\big) = -\lambda^2(\lambda - 2),$$

so the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$.
(b) For A to be diagonalizable the dimension of the Eigenspace corresponding to the double eigenvalue 0 has to be 2. We hence wish to solve $A\mathbf{x} = \mathbf{0}$. Row reduction gives the system $x_1 + x_3 = 0$ and $x_2 - x_3 = 0$, which can't be reduced further. Since the only free variable is x_3 the eigenspace is one dimensional. Therefore the matrix can not be diagonalized.

2. Let A be a 2×2 matrix with eigenvalues $1/2$ and $-1/2$.
Let $\operatorname{Ker}\big(A - \frac{1}{2}I\big) = \operatorname{Span}\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$ and $\operatorname{Ker}\big(A + \frac{1}{2}I\big) = \operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.
(a) Let $\mathbf{x}(t+1) = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = A\mathbf{x}(t)$. Given that $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $\mathbf{x}(3)$.
(b) Draw the phase portrait for the discrete system in part (a).
Sol. (a) We have $\mathbf{x}(k) = A^k\mathbf{x}(0)$. We want to write $\mathbf{x}(0) = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the eigenvectors corresponding to the eigenvalues $\lambda_1 = 1/2$ and $\lambda_2 = -1/2$ respectively. Then $\mathbf{x}(k) = A^k\mathbf{x}(0) = c_1A^k\mathbf{b}_1 + c_2A^k\mathbf{b}_2 = c_1\lambda_1^k\mathbf{b}_1 + c_2\lambda_2^k\mathbf{b}_2$.
Solving the system $\mathbf{x}(0)=c_1\mathbf{b}_1+c_2\mathbf{b}_2$ for c_1 and c_2 gives $c_1=0$ and $c_2=1$ so $\mathbf{x}(3)=(-1/2)^3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.
(a) Given that $\lambda = 1$ and 5 are the only eigenvalues of A. Find an orthonormal basis of $\mathbb{R}^3$ denoted by $\mathcal{B}$ consisting of eigenvectors of A.
(b) Given the following quadratic form $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 3x_2^2$.
Describe q in terms of $\mathcal{B}$ coordinates. Show work.
Sol.(a) $\operatorname{Ker}\big(A-I\big)=\operatorname{Span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ and $\operatorname{Ker}\big(A-5I\big)=\operatorname{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$, so $\mathbf{b}_1=\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are orthonormal.
(b) We have $A = QDQ^T$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} | & | \\ \mathbf{b}_1 & \mathbf{b}_2 \\ | & | \end{bmatrix}$. Set $\mathbf{y} = Q^T\mathbf{x}$.
Then $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x}\rangle = \langle \mathbf{x}, QDQ^T\mathbf{x}\rangle = \langle Q^T\mathbf{x}, DQ^T\mathbf{x}\rangle = \langle \mathbf{y}, D\mathbf{y}\rangle = y_1^2 + 5y_2^2 = \widetilde{q}(\mathbf{y})$.

8. Let A be a 2×2 matrix with eigenvalues 1 and 3,
such that $\text{Ker}\,\big(A - I\big) = \text{Span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ and $\text{Ker}\,\big(A - 3I\big) = \text{Span}\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\}$.
(a) Find A. Show work.
(b) Let T denote the transformation $T\mathbf{x} = A\mathbf{x}$. Write down the matrix of the transformation T with respect to the basis $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\}$. Show work.

Sol. (a) Since the eigenvalues are different A can be diagonalized $A = SDS^{-1}$, where $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ and $S = \begin{bmatrix} -1 & 1 \\ 1 & -3 \end{bmatrix}$ and hence $S^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$. Hence $A = \dots$.
(b) The matrix is the matrix D in (a).

10. State true or false with justification.
10(i) If A is a orthogonal 3×3 matrix then $\det A > 0$.
10(iv) If A is a 2×2 symmetric matrix then all its eigenvalues are positive real numbers.
Sol. (i) False, since Q could be a reflection, even just $-I$.
(iv) False, since again we could take $A = -I$.