May 7, 2009

Name ................
Section/ Name of your TA ................

Final Exam 200pts.
Math 201 Ver ****

• There are 12 pages in the exam excluding this page.
• Write all your answers clearly. You have to show work to get points for your answers.
• Read all the questions carefully and make sure you answer all the parts.
• Questions 1-8 have parts in them which are inter-related.
• You can write on both sides of the paper. Indicate that the answer follows on the back of the page.
• Use of Calculators is not allowed during the exam.

(1) .......... /20
(2) .......... /20
(3) .......... /20
(4) .......... /15
(5) .......... /15
(6) .......... /15
(7) .......... /20
(8) .......... /15
(9) .......... /32
(10) .......... /28

Total .......... /200
20pts. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of $A$.

(b) Is $A$ diagonalizable? explain why or why not?
Let $A$ be a $2 \times 2$ matrix with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$. Let

$$\text{Ker } (A - \frac{1}{2}I) = \text{Span}\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$$

and

$$\text{Ker } (A + \frac{1}{2}I) = \text{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}.$$ 

(a) Let

$$\vec{x}(t + 1) = \begin{bmatrix} x_1(t + 1) \\ x_2(t + 1) \end{bmatrix} = A\vec{x}(t).$$

Given that $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $\vec{x}(3)$.

(b) Draw the phase portrait for the discrete system in part (a).
3 20pts. Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

(a) Given that $\lambda = 0$ and 2 are the only eigenvalues of $A$. Find an orthonormal basis of $\mathbb{R}^3$ denoted by $\mathcal{B}$ consisting of eigenvectors of $A$.

(b) Given the following quadratic form $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 3x_2^2$. Describe $q$ in terms of $\mathcal{B}$ coordinates. Show work.
4 15pts. Let $f$ denote a infinitely differentiable function on $\mathbb{R}$. Find all real solutions to the following differential equation.

$$\frac{d^2 f}{dt^2} - f(t) = 0.$$
(a) Find the inverse of $A$, if it exists.

\[
A = \begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & -2 \\
3 & 1 & 0
\end{bmatrix}
\]

(b) Give a basis of the Image of the transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as $T(\vec{x}) = A\vec{x}$. 
(a) Given that the above is the augmented matrix of a system of equations, find $h$ such that it is consistent.

(b) For $h = 0$ find the least squares solution to the system.
Let \[ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \] and \[ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \] be two different bases of a subspace \( W \) in \( \mathbb{R}^3 \).

(a) Which of the two sets are orthogonal? Show work.

(b) Let \( \vec{y} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \). Is \( \vec{y} \in W \)?

(c) Find \( \text{proj}_W \vec{y} \), that is, the orthogonal projection of \( \vec{y} \) onto \( W \).
Let $A$ be a $2 \times 2$ matrix with eigenvalues 1 and 3, such that $\text{Ker}(A - I) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$ and $\text{Ker}(A - 3I) = \text{Span}\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$.

(a) Find $A$. Show work.

(b) Let $T$ denote the transformation $T\vec{x} = A\vec{x}$. Write down the matrix of the transformation $T$ with respect to the basis $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\right\}$. Show work.
9 32pts. Answer the following in short. **Give justification for your answers.**

(i) Let $\mathcal{D}$ denote the space of differentiable functions from $\mathbb{R} \to \mathbb{R}$. Is the function $<,>: \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ defined as

$$< f, g > = f(0)g'(0) + f'(0)g(0)$$

an inner product on $\mathcal{D}$?

(ii) Let $V = \text{Span}\{ \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\}$. Find the dimension of $V$. Explain your answer.
9(iii) Let $A$ be a $2 \times 2$ matrix with eigenvalues $-1 \pm 2i$. Then consider the system of differential equations,

$$\begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt}
\end{bmatrix} = A \begin{bmatrix} x_1 \\
 x_2
\end{bmatrix}.$$ 

What happens to $x(t)$ as $t \to \infty$? Show work.

9(iv) Let $A$ be an $2 \times 2$ matrix such that $A^3 = \begin{bmatrix} 1 & 0 \\
0 & 1
\end{bmatrix}$. Then find Ker $A$. 
10 28pts. State true or false with justification.

10(i) If $A$ is a orthogonal $3 \times 3$ matrix then $\det A > 0$.

10(ii) Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ and $\vec{w}_2 \in \text{Span}\{\vec{w}_1, \vec{w}_3\}$. Then $W = \text{Span}\{\vec{w}_1, \vec{w}_3\}$. 
10(iii) Let $T : V \rightarrow W$ be an invertible linear transformation from a vector space $V$ to another vector space $W$. If \{v_1, v_2, v_3\} is a linearly independent subset of $V$, then $\{Tv_1, Tv_2, Tv_3\}$ is a linearly independent set in $W$.

10(iv) If $A$ is a $2 \times 2$ symmetric matrix then all its eigenvalues are positive real numbers.