1. How many solutions are there to the system of linear equation
\[
\begin{align*}
&x_1 - 3x_2 = 0 \\
&3x_1 - 2x_2 = 7 \\
&2x_1 + x_2 = 7
\end{align*}
\]

**Sol.** The second equation is the sum of the first and third so it is not needed. The remaining 2x2 system
\[
\begin{bmatrix}
1 & -3 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
7
\end{bmatrix}
\]
has an invertible coefficient matrix so it is uniquely solvable.

2. The vectors \( v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and \( v_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \) form a basis for \( \mathbb{R}^2 \). Express the vector \( e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) as a linear combination of \( v_1 \) and \( v_2 \).

**Sol.** Since \( e_1 = c_1v_1 + c_2v_2 \) gives the system \( 2c_1 + 5c_2 = 1 \) and \( c_1 + 3c_2 = 0 \). The second equation gives \( c_1 = -3c_2 \) which if we substitute into the first gives \(-c_2 = 1 \) so \( c_2 = -1 \) and \( c_1 = 3 \).

3. Suppose a linear transformation \( T \) has the property that \( T(v_1) = v_1 + v_2 \) and \( T(v_2) = 2v_1 + 3v_2 \). Use your answer to problem 2 to find the value of \( T(e_1) \).

**Sol.** \( T(e_1) = T(3v_1 - v_2) = 3(v_1 + v_2) - (2v_1 + 3v_2) = v_1 \).

4. Are the vectors \( v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \), \( v_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \), \( v_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \), and \( v_4 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \) all linearly independent?

If not, identify which of these vectors are redundant.

**Sol.** They are linearly dependent since it is four vectors in a three dimensional space. The rest of the problem is most easily solved as in Problem 5.

5. Consider the matrix \( A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ -1 & -2 & 1 & 3 \\ 1 & 2 & 2 & 0 \end{bmatrix} \). Choose a basis for the image of \( A \).

**Sol.** Row reduction gives
\[
\begin{bmatrix}
2 & 4 & 3 & 1 \\
-1 & -2 & 1 & 3 \\
1 & 2 & 2 & 0
\end{bmatrix}
\Leftrightarrow
\begin{bmatrix}
0 & 0 & -1 & 1 \\
0 & 0 & 3 & -3 \\
1 & 2 & 2 & 0
\end{bmatrix}
\Leftrightarrow
\begin{bmatrix}
1 & 2 & 2 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\Leftrightarrow
\begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\Leftrightarrow
\begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix} - (2) \begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix} - (1) \begin{bmatrix}
1 & 2 & 0 & 2 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

The first and third column of \( B = \text{RREF}(A) \) corresponds to the leading variables and as a consequence the first and third column of \( A \) form a basis for the image of \( A \), see section 3.2-3.

**Rem.** This is because any linear relation amongst the columns of \( B \) corresponds to the same relation amongst the columns of \( A \) since \( Bx = 0 \) has the same solution set as \( Ax = 0 \), i.e. if \( A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}, \ B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \) then \( x_1a_1 + x_2a_2 + x_3a_3 + x_4a_4 = 0 \) if and only if \( x_1b_1 + x_2b_2 + x_3b_3 + x_4b_4 = 0 \). We have \( b_4 = 2b_1 - b_3 \) so \( a_4 = 2a_1 - a_3 \) and \( b_2 = 2b_1 \) so \( a_2 = 2a_1 \).

6. Chose a basis for the kernel of \( A \).

**Sol.** By the solution to problem 5, \( x_2 \) and \( x_4 \) are free variables so the and \( x_1 + 2x_2 + x_4 = 0 \) and \( x_3 - x_4 = 0 \) so the solution set is
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = x_2 \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
-1 \\
0 \\
1 \\
1
\end{bmatrix}
\]
and these vectors form a basis for the kernel.
7. Let \( Tf(x) = \frac{f(x) - f(0)}{x} \), acting on functions \( f \).
If a domain of \( T \) is \( \mathcal{P}_n = \{ \text{polynomials } f(x) = a_0 + a_1 x + \cdots = a_n x^n \} \),
then what is the image of \( T \)?
**Sol.** The image is polynomial of degree one less \( Tf = a_1 + 2a_2 x + \cdots + na_n x^{n-1} \).

8. Show that the kernel of \( T \) is the space \( \mathcal{P}_0 \) of constant functions.
**Sol.** \( Tf = 0 \) is equivalent to \( f = a_0 \).

9. What is the determinant of the matrix \( M = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix} \)?
**Sol.** Subtracting a multiple of the first row from the second and third rows gives
\[
\begin{vmatrix} 1 & 2 & 1 & 0 \\ -1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 5 \\ 1 & 2 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix} = 1 \cdot 5 \cdot 3 \cdot 2 = 30,
\]
since the determinant of a triangular matrix is the product of the diagonal elements.
**Rem.** Note that we only used the row operators that subtract off a multiple of another row from the row we change and these don’t change the determinant.
Note also that alternatively as a second step one could have expanded along the first column.

10. Give an example of \( 2 \times 2 \) matrices \( A \) and \( B \) where \( \det(A) + \det(B) \) is not equal to \( \det(A + B) \).
**Sol.** If \( A = I \) and \( B = -I \).

11. Find all eigenvalues of the matrix \( A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix} \).
**Sol.** Since \( A \) is triangular the eigenvalues are the diagonal entries \( \lambda_1 = \lambda_2 = 0 \) and \( \lambda_3 = 2 \).

12. For each of the eigenvalues of \( A \), find the associated eigenspace.
**Sol.** For \( \lambda_1 = \lambda_2 = 0 \) we get the system \( (A - 0I)x = 0 \), which is equivalent to \( x_3 = 0 \) and \( x_2 = 0 \) and hence the eigenspace is one dimensional \( \text{Span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} \).
For \( \lambda_3 = 2 \) we get the system \( (A - 2I)x = 0 \), which is equivalent to \( -2x_1 + x_2 + 2x_2 = 0 \) and \( -x_2 + 2x_3 = 0 \) and hence the eigenspace is one dimensional \( \text{Span}\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \} \).

13. Is it possible to diagonalize the matrix \( A \)?
**Sol.** \( A \) is not diagonalizable since it does not have three linearly independent eigenvectors.

14. What is the length of the vector \( v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \)?
15. What is the angle between the vectors \( \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \) and \( \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \)?

Sol. \( \|\mathbf{v}\| \|\mathbf{e}_1\| \cos \theta = \mathbf{v} \cdot \mathbf{e}_2 = 1 \) so \( \cos \theta = \frac{1}{\sqrt{4}} \).

16. What is the projection of \( \mathbf{e}_2 \) onto the line spanned by \( \mathbf{v} \)?

Sol. Let \( \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \). Then the projection is \( (\mathbf{e}_2 \cdot \mathbf{u}) \mathbf{u} = \frac{\mathbf{e}_2 \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \).

17. The system of equations \( \begin{cases} x_1 = 5 \\ x_1 = 1 \text{ (equivalently, } \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}) \end{cases} \) is hopelessly inconsistent. What values of \( x_1 \) provides the least-squares approximate solution?

18. Decide whether the function \( \langle f, g \rangle = \int_{-1}^{1} f(x)g(-x)dx \) is a valid inner product, where \( f \) and \( g \) are allowed to be any pair of continuous functions on the interval \([-1, 1]\).

Sol. It is not since we can take \( f(t) = 0 \) when \( t < 0 \) but \( f(t) = t \) for \( t > 0 \) in which case \( \langle f, f \rangle = 0 \) but \( f \neq 0 \).

19. True or False: If \( A \) and \( B \) are both symmetric matrices, then their product \( AB \) must also be symmetric. Explain the reasoning behind your answer.

Sol. False, take e.g. \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

20. True or False: If \( A \) and \( B \) are both orthogonal matrices, then their product \( AB \) must also be orthogonal. Explain the reasoning behind your answer.

Sol. True, since \( A \) and \( B \) are orthogonal \( A^T A = I \) and \( B^T B = I \), and it follows that \( (AB)^T AB = B^T A^T A B = B^T I B = B^T B = I \) so \( AB \) is orthogonal.

21. How many complex eigenvalues does the matrix \( M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \) have?

Sol. Since \( M \) is symmetric all the eigenvalues are real.

22. Express the quadratic form \( q(x_1, x_2) = x_1^2 + 6x_1x_2 + 8x_2^2 \) as an inner product \( q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle \), where \( A \) is a symmetric matrix.

Sol. \( A = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix} \).

23. Is there a choice of numbers \( (x_1, x_2) \) for which \( q(x_1, x_2) \) is negative? What does the set of points where \( q(x_1, x_2) = 1 \) look like? [Please describe the overall shape of the set - it is not necessary to give exact specifications.]
The characteristic polynomial is \((1-\lambda)(8-\lambda) - 9 = \lambda^2 - 9\lambda - 1\) which has roots 
\(\lambda_\pm = 9/2 \pm \sqrt{(9/2)^2 + 1}\) so \(\lambda_- < 0\) and \(\lambda_+ > 0\). Since \(A\) is symmetric it can be diagonalized 
\(A = QDQ^T\) and if we set \(y = Q^T x\) we get 
\[ q(x) = \langle x, Ax \rangle = \langle x, QDQ^T x \rangle = \langle Q^T x, DQ^T x \rangle = \langle y, Dy \rangle = \lambda_- y_1^2 + \lambda_+ y_2^2 = \tilde{q}(y). \]
Hence \(\tilde{q}(1,0) = \lambda_- \leq 0\). Now, \(y_0 = (y_1, y_2) = (1, 0)\) corresponds to some \(x_0 = Qy_0\) such that 
\[ q(x_0) = \tilde{q}(y_0) = \lambda_- < 0. \] The set \(\tilde{q}(y) = \lambda_- y_1^2 + \lambda_+ y_2^2 = 1\) is a hyperbola.

24. What are the singular values of matrix 
\(A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \)?

**Sol.** The singular values are the square root of the eigenvalues of 
\(A^T A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \).

The characteristic polynomial is \((6-\lambda)(3-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda-9/2)^2 - 25/4\), so the eigenvalues are \(9/2 \pm 5/2\) so \(\lambda_1 = 7\) and \(\lambda_2 = 2\) and the singular values are 
\(\sigma_1 = \sqrt{7}\) and \(\sigma_2 = \sqrt{2}\).

25. Find a set of perpendicular vectors \(v_1\) and \(v_2\) in \(\mathbb{R}^2\) which have the additional property that \(Av_1\) and \(Av_2\) are also perpendicular to each other?

**Sol.** Let \(v_1\) and \(v_2\) be the normalized eigenvectors of \(A^T A\):
\[(A^T A - 7I)v_1 = 0 \text{ gives } v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } (A^T A - 2I)v_2 = 0 \text{ gives } v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.\]

We claim that \(Av_1\) and \(Av_2\) are perpendicular. In fact, 
\[ \langle Av_i, Av_j \rangle = \langle A^T Av_i, v_j \rangle = \langle \lambda_i v_i, v_j \rangle = \lambda_i \langle v_i, v_j \rangle, \]
and if \(i \neq j\) then \(\langle v_i, v_j \rangle = 0\).

**Remark** If \(i = j\) the above equation reads \(\|Av_i\|^2 = \lambda_i \|Av_i\|^2\), so the vectors \(u_i = Av_i/\sigma_i\), 
\(i = 1, 2\), are orthonormal. We have 
\[ u_1 = \frac{1}{\sqrt{35}} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{14}} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \] \(u_3 = u_1 \times u_2 = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.\)

The vectors can be used to obtain the singular value decomposition 
\(A = U \Sigma V^T\), where
\[ V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, \quad U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{35}} & \frac{-1}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{-2}{\sqrt{14}} \\ \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{14}} \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{bmatrix}. \]