11. Find all eigenvalues of the matrix \( A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix} \).

**Sol.** Since \( A \) is triangular the eigenvalues are the diagonal entries \( \lambda_1 = \lambda_2 = 0 \) and \( \lambda_3 = 2 \).

12. For each of the eigenvalues of \( A \), find the associated eigenspace.

**Sol.** For \( \lambda_1 = \lambda_2 = 0 \) we get the system \((A - 0I)x = 0\), which is equivalent to \(x_3 = 0\) and \(x_2 = 0\) and hence the eigenspace is one dimensional \( \text{Span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \} \).

For \( \lambda_3 = 2 \) we get the system \((A - 2I)x = 0\), which is equivalent to \(-2x_1 + x_2 + 2x_2 = 0\) and \(-x_2 + 2x_3 = 0\) and hence the eigenspace is one dimensional \( \text{Span}\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \} \).

13. Is it possible to diagonalize the matrix \( A \)?

**Sol.** \( A \) is not diagonalizable since it does not have three linearly independent eigenvectors.

19. True or False: If \( A \) and \( B \) are both symmetric matrices, then their product \( AB \) must also be symmetric. Explain the reasoning behind your answer.

**Sol.** False, take e.g. \( A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \).

20. True or False: If \( A \) and \( B \) are both orthogonal matrices, then their product \( AB \) must also be orthogonal. Explain the reasoning behind your answer.

**Sol.** True, since \( A \) and \( B \) are orthogonal \( A^T A = I \) and \( B^T B = I \), and it follows that \((AB)^T AB = B^T A^T AB = B^T IB = B^T B = I \) so \( AB \) is orthogonal.

21. How many complex eigenvalues does the matrix \( M = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \) have?

**Sol.** Since \( M \) is symmetric all the eigenvalues are real.
22. Express the quadratic form \(q(x_1, x_2) = x_1^2 + 6x_1x_2 + 8x_2^2\) as an inner product \(q(x) = \langle x, Ax \rangle\), where \(A\) is a symmetric matrix.

**Sol.** \(A = \begin{bmatrix} 1 & 3 \\ 3 & 8 \end{bmatrix}\).

23. Is there a choice of numbers \((x_1, x_2)\) for which \(q(x_1, x_2)\) is negative? What does the set of points where \(q(x_1, x_2) = 1\) look like? [Please describe the overall shape of the set - it is not necessary to give exact specifications.]

**Sol.** The characteristic polynomial is \((1 - \lambda)(8 - \lambda) - 9 = \lambda^2 - 9\lambda - 1\) which has roots \(\lambda_{\pm} = 9/2 \pm \sqrt{(9/2)^2 + 1}\) so \(\lambda_{\pm} < 0\) and \(\lambda_{\pm} > 0\). Since \(A\) is symmetric it can be diagonalized \(A = QDQ^T\) and if we set \(y = Q^T x\) we get

\[q(x) = \langle x, Ax \rangle = \langle x, QDQ^T x \rangle = \langle Q^T x, DQ^T x \rangle = \langle y, Dy \rangle = \lambda_1 y_1^2 + \lambda_2 y_2^2 = \tilde{q}(y).\]

Hence \(\tilde{q}(1, 0) = \lambda_1 < 0\). Now, \(y_0 = (y_1, y_2) = (1, 0)\) corresponds to some \(x_0 = Qy_0\) such that \(q(x_0) = \tilde{q}(y_0) = \lambda_1 < 0\). The set \(\tilde{q}(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 < 1\) is a hyperbola.

24. What are the singular values of matrix \(A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}\)?

**Sol.** The singular values are the square root of the eigenvalues of \(A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}\).

The characteristic polynomial is \((6-\lambda)(3-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = (\lambda-9/2)^2 - 25/4\), so the eigenvalues are \(9/2 \pm 5/2\) so \(\lambda_1 = 7\) and \(\lambda_2 = 2\) and the singular values are \(\sigma_1 = \sqrt{\pi}\) and \(\sigma_2 = \sqrt{2}\).

25. Find a set of perpendicular vectors \(v_1\) and \(v_2\) in \(\mathbb{R}^2\) which have the additional property that \(Av_1\) and \(Av_2\) are also perpendicular to each other?

**Sol.** Let \(v_1\) and \(v_2\) be the normalized eigenvectors of \(A^TA\):

\[(A^T A - 7I)v_1 = 0 \text{ gives } v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } (A^T A - 2I)v_2 = 0 \text{ gives } v_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.\]

We claim that \(Av_1\) and \(Av_2\) are perpendicular. In fact,

\[\langle Av_i, Av_j \rangle = \langle A^T A v_i, v_j \rangle = \langle \lambda_i v_i, v_j \rangle = \lambda_i \langle v_i, v_j \rangle,\]

and if \(i \neq j\) then \(\langle v_i, v_j \rangle = 0\).

**Remark** If \(i = j\) the above equation reads \(\|Av_i\|^2 = \lambda_i \|Av_i\|^2\), so the vectors \(u_i = Av_i/\sigma_i\), \(i = 1, 2\), are orthonormal. We have \(u_1 = \frac{1}{\sqrt{35}} \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, u_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}\) and \(u_3 = u_1 \times u_2 = \frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\).

The vectors can be used to obtain the singular value decomposition \(A = USV^T\), where

\[V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{35}} & \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{11}} \\ \frac{5}{\sqrt{35}} & 0 & \frac{-2}{\sqrt{11}} \\ \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{11}} \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{bmatrix}.\]