Name: ______________________________ May 12, 2006

Section Number/Time ______________________

TA ______________________

MATH 201, FINAL EXAM

Directions: This is a pencil-and-paper exam. You are asked to put away all books, notes, calculators, cell phones, and other computing and/or telecommunications equipment.

Grading: There are twenty-five questions on the exam, each worth 4 points, for a total of 100 points.

1. How many solutions are there to the system of linear equations $\begin{cases} x_1 - 3x_2 = 0 \\ 3x_1 - 2x_2 = 7 \\ 2x_1 + x_2 = 7 \end{cases}$

2. The vectors $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ form a basis for $\mathbb{R}^2$. Express the vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

3. Suppose a linear transformation T has the properties that $T(\vec{v}_1) = \vec{v}_1 + \vec{v}_2$ and $T(\vec{v}_2) = 2\vec{v}_1 + 3\vec{v}_2$. Use your answer from Problem 2 to find the value of $T(\vec{e}_1)$.

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4. Are the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$, and $\vec{v}_4 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$ all linearly independent?

If not, identify which of these vectors are redundant.

5. Consider the matrix $A = \begin{bmatrix} 2 & 4 & 3 & 1 \\ -1 & -2 & 1 & -3 \\ 1 & 2 & 2 & 0 \end{bmatrix}$. Choose a basis for the image of A.

6. Choose a basis for the kernel of A.

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Problems 7 and 8 deal with the linear transformation $Tf(x) = \dfrac{f(x) - f(0)}{x}$, acting on functions f.

7. If the domain of T is $\mathcal{P}_n = \{\text{polynomials } f(x) = a_0 + a_1 x + \ldots + a_n x^n\}$, then what is the image of T?

8. Show that the kernel of T is the space $\mathcal{P}_0$ of constant functions.

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9. What is the determinant of the matrix $M = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix}$?

10. Give an example of 2×2 matrices A and B where $\det(A) + \det(B)$ is not equal to $\det(A + B)$.

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11. Find all the eigenvalues of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$.

12. For each of the eigenvalues of A, find the associated eigenspace.

13. Is it possible to diagonalize the matrix A?

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14. What is the length of the vector $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$?

15. What is the angle between vectors $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$?

16. What is the projection of $\vec{e}_2$ onto the line spanned by $\vec{v}$?

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17. The system of equations $\begin{cases} x_1 = 5 \\ x_1 = 1 \\ x_1 = 6 \end{cases}$ $\left(\text{equivalently, } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_1 = \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix}\right)$ is hopelessly inconsistent.

What value of x_1 provides the least-squares approximate solution?

18. Decide whether the function $\langle f, g \rangle = \int_{-1}^{1} f(x)g(-x)\,dx$ is a valid inner product, where f and g are allowed to be any pair of continuous functions over the interval $[-1, 1]$.

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19. True or False:

If A and B are both symmetric matrices, then their product AB must also be symmetric.
Explain the reasoning behind your answer.

20. True or False:

If A and B are both orthogonal matrices, then their product AB must also be orthogonal.
Explain the reasoning behind your answer.

21. How many complex eigenvalues does the matrix $M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 3 & 3 \\ 1 & 2 & 3 & 0 & 4 \\ 1 & 2 & 3 & 4 & 0 \end{bmatrix}$ have?

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22. Express the quadratic form $q(x_1, x_2) = x_1^2 + 6x_1x_2 + 8x_2^2$ as an inner product $q(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle$, where A is a symmetric matrix.

23. Is there any choice of numbers (x_1, x_2) for which $q(x_1, x_2)$ is negative?

What does the set of points where $q(x, y) = 1$ look like?

[Please describe the overall shape of this set – it is not necessary to give its exact specifications.]

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24. What are the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$?

25. Find a set of perpendicular vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$ which have the additional property that $A\vec{v}_1$ and $A\vec{v}_2$ are also perpendicular to each other.