5a. [2 marks] Find the eigenvalues of \( A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \).

5b. [3 marks] Give a factorization \( A = QDQ^T \) where \( Q \) has orthonormal columns and \( D \) is a diagonal matrix.

5d. [1 marks] Is \( A \) a positive definite matrix? Why? Give the quadratic form \( q(x, y) \) associated to \( A \).

Sol.

5a. Solve \( \det(A - \lambda I) = 0 \). We get the equation \( \lambda(\lambda - 5) = 0 \). Hence \( \lambda_1 = 5 \) and \( \lambda_2 = 0 \).

5b. \( A = QDQ^T \) is an “eigenvalue-eigenvector” factorization of a symmetric matrix. \( D \) is a diagonal matrix containing the eigenvalues of \( A \) and \( Q \) is a \( 2 \times 2 \)-matrix whose orthonormal columns are eigenvectors of \( A \). For example

\[
D = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.
\]

5d. \( A \) is not a positive definite matrix as \( \lambda_2 = 0 \). The quadratic form (singular) associated to \( A \) is \( q(x, y) = x^2 + 4xy + 4y^2 \).

6a. [3 marks] If possible, find an invertible matrix \( M \) such that

\[
M^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}.
\]

If it is not possible, state why \( M \) cannot exist.

6b. [3 marks] For what real values of \( c \) (if any) is \( A = \begin{bmatrix} -1 & c & 2 \\ c & -4 & -3 \\ 2 & -3 & 4 \end{bmatrix} \) a symmetric positive definite matrix?

6c. [4 marks] Let \( A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \). Is the quadratic form \( q(x, y) \) associated to \( A \) positive definite? Find its principal axes.

Sol.

6a. Not possible. The condition means that \( B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix} \) is similar to \( A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \).

Similar matrices have equal traces and rank. But \( \text{trace}(A) = 3 \neq \text{trace}(B) = 5 \) and \( \text{rk}(A) = 1 \neq \text{rk}(B) = 2 \).

6b. Not possible. For symmetric positive-definite matrices all upper-left determinants are greater than zero. Note that the 1 by 1 upper-left determinant is \(-1\).

6c. \( \det(A) < 0 \), hence \( A \) (or equivalently its associated quadratic form \( q(x, y) = 3x^2 + 8xy + 3y^2 \)) is not positive definite. The eigenvalues of \( A \) are: \( \lambda_1 = -1 \) and \( \lambda_2 = 7 \). The principal axes are the eigenspaces of \( A \), namely \( E_1 = \text{span}\{\begin{bmatrix} -1 \\ 0 \end{bmatrix}\} \) and \( E_2 = \text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\} \).
7a. [6 marks] Let $A_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Is $A_1$ diagonalizable? Why? Is $A_1$ invertible? Why? Determine the spectral decomposition of $A_1$ into projection matrices.

7b. [3 marks] Let $A_2 = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$. Is $A_2$ invertible? Why? Is $A_2$ diagonalizable? Why? Determine (if exists) a matrix $S$ and a diagonal matrix $D$ such that $S^{-1}A_2S = D$.

7c. [6 marks] Describe the linear transformation $T_{A_2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to $A_2$. Does such $A_2$ have a decomposition into projection matrices? If yes, give it.

Sol.

7a. $A_1$ is symmetric, hence $A_1$ is diagonalizable. $A_1$ is invertible as $\det(A_1) = \prod \lambda_i = 2 \cdot 1 \cdot (-1) = -2 \neq 0$. The spectral decomposition of $A_1$ is given by

$$A_1 = \sum_{i=1}^{3} \lambda_i x_i x_i^T$$

where $x_i$ are eigenvectors associated to the eigenvalues $\lambda_i$. We can choose $x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ as eigenvector associated to $\lambda_1 = 2$, $x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ as eigenvector associated to $\lambda_2 = 1$ and $x_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as eigenvector associated to $\lambda_3 = -1$. It follows that the spectral decomposition of $A_1$ is

$$A_1 = 2P_1 + P_2 - P_3 = 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7b. $\det(A_2) = 0$ hence $A_2$ is not invertible. The eigenvalues of $A_2$ are $\mu_1 = 0$ and $\mu_2 = -4$. They are distinct, hence $A_2$ is diagonalizable. The columns of the matrix $S$ are made by 2 eigenvectors of $A$ i.e. $S = \begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}$, whereas $D = \begin{bmatrix} 0 & 0 \\ 0 & -4 \end{bmatrix}$ is the eigenvalues matrix.

7c. The linear transformation $T_{A_2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is determined by a projection $P_2$ onto the line spanned by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$A_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0, \quad A_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$ 

It follows that

$$A_2 = -4 \begin{bmatrix} 3/4 & -3/4 \\ -1/4 & 1/4 \end{bmatrix} = -4P_2$$

is the required decomposition, where $P_2$ is a projection matrix.
8a. [3 marks] Find the lengths and the inner product $\vec{x} \cdot \vec{y}$ of the following complex vectors

$$\vec{x} = \begin{bmatrix} 2 - 4i \\ 4i \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad (i^2 = -1).$$

8b. [3 marks] Let $A = \begin{bmatrix} 1 & 1 - i \\ 1 + i & 2 \end{bmatrix}$. Let $\vec{x}_1, \vec{x}_2$ be two (linearly independent) eigenvectors of $A$. Compute $\vec{x}_1 \cdot \vec{x}_2$ and show that $\det(A) \in \mathbb{R}$.

8c. [4 marks] Prove that for any complex vector $\vec{x}$

$$\vec{x}^H A \vec{x} \in \mathbb{R}. \quad (H = \text{Hermitian})$$

Sol.

8a. length($\vec{x}$) = $(\vec{x}^H \vec{x})^{1/2} = (2 + 4i \cdot 4i)^{1/2} = 6$; length($\vec{y}$) = $(\vec{y}^T \vec{y})^{1/2} = \sqrt{20}$

and $\vec{x} \cdot \vec{y} := \vec{x}^H \vec{y} = 4(1 - 2i)$.

8b. Notice that $A = A^H$, furthermore let $\lambda_i$ be the 2 eigenvalues of $A$: $0 = \lambda_1 \neq \lambda_2 = 3$, then $\vec{x}_1 \cdot \vec{x}_2 = 0$. Also, one knows that every eigenvalue of a Hermitian matrix is real and so will be its determinant ($\det(A) = \lambda_1 \lambda_2 = 0$).

8c. We have $(\vec{x}^H A \vec{x})^H = \vec{x}^H A \vec{x}$, as $A = A^H$. It follows that $\vec{x}^H A \vec{x} \in \mathbb{R}$. 