THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
FINAL EXAM - SPRING SESSION 2005
110.201 - LINEAR ALGEBRA.

Examiner: Professor C. Consani
Duration: 3 HOURS (9am-12noon), May 12, 2005.

No calculators allowed. **Total Marks = 100**

Student Name: ______________________________

TA Name & Session: ________________________

1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	
Total	

1. [10 marks] Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 2 & 1 & 1 & 2 \\ 4 & 2 & 3 & 2 \end{bmatrix}$.

1a. [2 marks] Compute the reduced row-echelon form of A.

1b. [2 marks] Determine the rank of A.

1c. [2 marks] Determine a basis of the column space of A.

1d. [2 marks] Determine a basis of the nullspace of A.

1e. [2 marks] For what value(s) of $r \in \mathbb{R}$ is the following system solvable

$$A\underline{x} = \begin{bmatrix} 2 \\ 3 \\ r \end{bmatrix}?$$

2. [15 marks] Consider the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

2a. [5 marks] Give a factorization $A = QR$, where R is an upper-triangular matrix and Q is a matrix with orthonormal columns.

2b. [5 marks] Find the least square solution to the system

$$A\underline{x} = \underline{b}, \qquad \text{for} \quad \underline{b} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}.$$

2c. [5 marks] The projection matrix $P = A(A^TA)^{-1}A^T$ projects all vectors onto the column space of A. Find a vector $\underline{q}$, <u>not</u> in the column space of A such that

$$P\underline{q} = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}.$$

3. [15 marks]

3a. [4 marks] Give a 3×3-matrix A with the following properties:

i. $A^T = A^{-1}$.
ii. $\det(A) = 1$. (A is not allowed to be a diagonal matrix)

3b. [4 marks] Give a 3×3-matrix with the following properties:

i. $A^T = A$.
ii. $A^2 = A$.
iii. $\text{rk}(A) = 1$. (A is not allowed to be a diagonal matrix)

3c. [4 marks] Suppose A is a 5×3-matrix with orthonormal columns. Evaluate the following determinants:

i $\det(A^T A)$
ii $\det(AA^T)$
iii $\det(A(A^T A)^{-1} A^T)$.

3d. [3 marks] Which value(s) of $\alpha \in \mathbb{R}$ give $\det(A) = 0$, if

$$A = \begin{bmatrix} \alpha & 2 & 3 \\ -\alpha & \alpha & 0 \\ 3 & 2 & 5 \end{bmatrix}?$$

4. [15 marks] Suppose the following information is known about a matrix A:

i. $A \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}$

ii. $A \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}$

iii. A is symmetric.

The following questions refer to <u>any</u> matrix A with the above properties

4a. [3 marks] Is $\text{Ker}(A) = \{\underline{0}\}$? Explain your answer.

4b. [3 marks] Is A invertible? Why?

4c. [3 marks] Does A have linearly independent eigenvectors? Explain.

4d. [6 marks] Give a specific example of a matrix A satisfying the above three properties and whose eigenvalues add up to zero.

5. [10 marks] Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

5a. [2 marks] Find the eigenvalues of A.

5b. [3 marks] Give a factorization $A = QDQ^T$ where Q has orthonormal columns and D is a diagonal matrix.

5c. [4 marks] As $t \to \infty$, what is the limit of $\underline{u}(t)$ for

$$\frac{d\underline{u}(t)}{dt} = -A\underline{u}(t)$$

given the initial condition $\underline{u}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

5d. [1 marks] Is A a positive definite matrix? Why? Give the quadratic form $q(x, y)$ associated to A.

6. [10 marks]

6a. [3 marks] If possible, find an invertible matrix M such that

$$M^{-1}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}M=\begin{bmatrix}1&1&1\\1&2&2\\1&2&2\end{bmatrix}.$$

If it is not possible, state why M cannot exist.

6b. [3 marks] For what real values of c (if any) is

$$A=\begin{bmatrix}-1&c&2\\c&-4&-3\\2&-3&4\end{bmatrix}$$

a symmetric positive definite matrix?

6c. [4 marks] Let $A=\begin{bmatrix}3&4\\4&3\end{bmatrix}$. Is the quadratic form $q(x,y)$ associated to A positive definite? Find its principal axes.

7. [15 marks]

7a. [6 marks] Let $A_1 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Is A_1 diagonalizable? Why? Is A_1 invertible? Why? Determine the spectral decomposition of A_1 into projection matrices.

7b. [3 marks] Let $A_2 = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$. Is A_2 invertible? Why? Is A_2 diagonalizable? Why? Determine (if exists) a matrix S and a diagonal matrix D such that $S^{-1}A_2S = D$.

7c. [6 marks] Describe the linear transformation $T_{A_2} : \mathbb{R}^2 \to \mathbb{R}^2$ associated to A_2. Does such A_2 have a decomposition into projection matrices? If yes, give it.

8. [10 marks]

8a. [3 marks] Find the lengths and the inner product $\underline{x} \cdot \underline{y}$ of the following complex vectors

$$\underline{x} = \begin{bmatrix} 2-4i \\ 4i \end{bmatrix}, \qquad \underline{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad (i^2 = -1).$$

8b. [3 marks] Let $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$. Let $\underline{x}_1$, $\underline{x}_2$ be two (linearly independent) eigenvectors of A. Compute $\underline{x}_1 \cdot \underline{x}_2$ and show that $\det(A) \in \mathbb{R}$.

8c. [4 marks] Prove that for any complex vector $\underline{x}$

$$\underline{x}^H A \underline{x} \in \mathbb{R}. \qquad (H = \text{Hermitian})$$