3. (a) Let \( A = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix} \). Determine a diagonal matrix \( D \), and an orthogonal matrix \( S \) for which \( A = SDS^{-1} \). Multiply out \( SDS^{-1} \) to check that your answer is correct.

(b) Let \( B = \begin{bmatrix} 4 & 3 \\ -3 & -4 \end{bmatrix} \). Determine whether \( B \) is similar to the matrix \( A \) from part (a) in \( \mathbb{R}^{2 \times 2} \).

**Sol.** (a) The characteristic polynomial is \((4-\lambda)(-4-\lambda) - 9 = \lambda^2 - 25 = (\lambda - 5)(\lambda + 5)\). We have \( \text{Ker} (A+5I) = \text{Span}\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \} \) and \( \text{Ker} (A-5I) = \text{Span}\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \} \).

Let \( S = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}, D = \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix} \).

(b) The characteristic polynomial is \((4-\lambda)(-4-\lambda) + 9 = \lambda^2 - 7 = (\lambda - \sqrt{7})(\lambda + \sqrt{7})\). Since similar matrices have the same eigenvalues \( B \) can not be similar to \( A \).

5. (a) Let \( V \) be a linear space. Suppose that \( \lambda \) is an eigenvalue of the linear transformation \( T : V \to V \). Derive the fact that \( \lambda^2 \) is an eigenvalue of \( T^2 \).

(b) Determine all matrices in \( \mathbb{R}^{3 \times 3} \) that are both symmetric and orthogonal, and describe them geometrically. [Suggestion: Express the two conditions in terms of 'transpose'.]

**Sol.** (a) Since \( \lambda \) is an eigenvalue there is a \( v \neq 0 \) such that \( Tv = \lambda v \). Hence \( T^2v = T(T(v)) = T(\lambda v) = \lambda T(v) = \lambda^2 v \) which proves that \( \lambda^2 \) is an eigenvalue for \( T^2 \).

(b) \( A^T A = I \) and \( A^T = A \) so \( A^2 = I \). Moreover \( A \) is diagonalizable so \( A = QDQ^T \), where \( D \) is diagonal and \( Q^TQ = QQ^T = I \). Hence \( A^2 = QDQ^T DQ^T = QD^2 Q^T = I \) so \( D^2 = Q^TIQ = I \). It follows that the eigenvalues of \( A \) are all \(-1\) or \(1\). On the other if \( D \) is diagonal with \( \pm 1 \) in the diagonals then \( A = QDQ^T \) then \( A^2 = QDQ^T DQ^T = D^2 = Q^TIQ = I \).

7. For which \( a \in \mathbb{R}, b \in \mathbb{R} \) does the matrix \( A = \begin{bmatrix} 2 & 0 & 0 \\ b & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \) have an eigenbasis (for \( \mathbb{R}^3 \))?

When it does, specify an eigenbasis (depending on \( a \) and \( b \)).

**Sol.** Since the matrix is triangular the eigenvalues are the diagonal elements \(1\) and \(2\). \((A - I)x = 0\) is equivalent to \( x_1 = 0 \) and \( ax_2 = 0 \).

Hence if \( a \neq 0 \) \( \text{Ker} (A - I) = \text{Span}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \} \), and if \( a = 0 \) \( \text{Ker} (A - I) = \text{Span}\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} \).

\((A - 2I)x = 0\) is equivalent to \( bx_1 - x_2 = 0 \) and \( ax_2 - x_3 = 0 \).

\( \text{Ker} (A - 2I) = \text{Span}\{ \begin{bmatrix} 1 \\ b \\ ab \end{bmatrix} \} \).

Hence \( A \) has an eigenbasis only if \( a = 0 \).
8. Let $V$ be $\text{Span}\{1, \sin x, \cos x\}$. The dimension of $V$ is 3.
(c) Let $D$ denote the linear operator on $V$ given by $D(f) = f'$. Determine the complex eigenvalues of $D$—that includes the real ones!—and the corresponding eigenspaces.

**Sol.** The matrix is $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$. The characteristic polynomial is $\lambda(\lambda^2 + 1) = \lambda(\lambda + i)(\lambda - i)$.

\[
\text{Ker}(A - iI) = \text{Span}\{i\}, \quad \text{Ker}(A + iI) = \text{Span}\{-i\}, \quad \text{Ker}(A - 0I) = \text{Span}\{1\},
\]

so the eigenvectors are $i \sin x + \cos x$ and $-i \sin x + \cos x$ and 1.

9. Let $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$ where $b$ and $c$ are real scalars. Determine the set of values of $b$ and $c$ for which the dynamical system $\mathbf{x}(t+1) = A\mathbf{x}(t)$ is asymptotically stable
(meaning: for all initial states, the state vector tends to 0, as $t \to \infty$).

**Sol.** The characteristic polynomial is $(1 - \lambda)^2 - bc = (\lambda - 1 - \sqrt{bc})(\lambda - 1 + \sqrt{bc})$.

If $bc > 0$ the eigenvalues are $\lambda = 1 \pm \sqrt{bc}$, if $bc < 0$ then $\lambda = 1 \pm i\sqrt{bc}$ and if $bc = 0$ $\lambda = 1$.

If $bc \neq 0$ the eigenvalues are distinct and therefore we have a basis of eigenvectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

If $bc \neq 0$ we can therefore write $\mathbf{x}(0) = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2$. It follows that $\mathbf{x}(k) = A^k \mathbf{x}(0) = c_1 A^k \mathbf{b}_1 + c_2 A^k \mathbf{b}_2 = c_1 \lambda_1^k \mathbf{b}_1 + c_2 \lambda_2^k \mathbf{b}_2$. Hence $\mathbf{x}(k) \to 0$ as $k \to \infty$ only if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

If $bc \neq 0$ at least one eigenvalue satisfy $|\lambda| \geq 1$ so it is not asymptotically stable.

If $b = c = 0$ the matrix is the identity so the eigenvalues are both 0 and it is not stable.

If $c = 0$ but $b \neq 0$ (or the other way around) then we have at least one eigenvector $\mathbf{b}_1$ with eigenvalue $\lambda_1 = 1$ so if the solution initially is in the state i.e. $\mathbf{x}(0) = c_1 \mathbf{b}_1$, with $c_1 \neq 0$ then $\mathbf{x}(k) = c_1 \mathbf{b}_1$, for all $k$ which does not tend to 0 as $k \to \infty$. Hence the system is not stable.

**Rem** If $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ then we do not have basis of eigenvectors so we can not use this method.

It is, however, easy to see that $A^k = \begin{bmatrix} 1 & kb \\ 0 & 1 \end{bmatrix}$.

10. Determine whether $q(x_1, x_2) = x_1^2 + 3x_1x_2 + 2x_2^2 = 1$ is the equation of an ellipse.

**Sol.** $q(\mathbf{x}) = \langle \mathbf{x}, A\mathbf{x} \rangle$, where $A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 2 \end{bmatrix}$. The characteristic polynomial is $(1 - \lambda)(2 - \lambda) - 9/4 = \lambda^2 - 3\lambda + 2 - 9/4 = (\lambda - 3/2)^2 - 5/2 < 0$ and $\lambda_2 = 3/2 + \sqrt{5/2} > 0$. Since $A$ is symmetric we can diagonalize $A = QDQ^T$ and we get $q(\mathbf{x}) = \langle \mathbf{x}, QDQ^T \mathbf{x} \rangle = \langle Q^T \mathbf{x}, DQ^T \mathbf{x} \rangle = \langle \mathbf{y}, D\mathbf{y} \rangle = \tilde{q}(\mathbf{y})$, where $\mathbf{y} = Q^T \mathbf{x}$. Hence $q(\mathbf{x}) = \tilde{q}(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$ is not an ellipse in the y coordinates, which is just a rotation or reflection of the x coordinates.

11. (a) Give an example of a $2 \times 2$ real matrices that have the same characteristic polynomial yet they are not similar. Explain.

(b) True or False: If a matrix fails to diagonalize over $\mathbb{R}$, it will diagonalize over $\mathbb{C}$. Explain.

**Sol.** (a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. (b) False, e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.