ETHICS PLEDGE: I agree to complete this exam without unauthorized assistance from any person or person’s work, materials or device.

Your name (print): ____________________________  Section: ________

Signature: ________________________________  Date: ________
No books, no notes, no calculators or other electronic devices. Write legibly, and show all relevant work—or risk losing credit. Answer what is asked, and only what is asked.

[30] 1. Let \( A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & -1 & -2 \end{bmatrix} \).

a) Give the definition of \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is a *linearly independent set* in terms of linear combinations or linear relations. Use this definition to show that \( \mathcal{B} = \{ \vec{e}_3, \vec{e}_1 + \vec{e}_2, \vec{e}_1 - \vec{e}_2 \} \) is a basis of \( \mathbb{R}^3 \).

b) Determine the \( \mathcal{B} \)-matrix of \( A \), where \( \mathcal{B} \) is the basis in part a).

c) Determine all real numbers \( c \) for which the equation (for \( \vec{x} \in \mathbb{R}^3 \)) \( A\vec{x} = c\vec{x} \) has a non-trivial solution.
[20] 2. Let \( A = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{bmatrix} \).

a) Determine all vectors \( \vec{x} \) that satisfy the linear system \( A\vec{x} = \vec{0} \).

b) Explain why \( W = \{ \vec{b} \in \mathbb{R}^4 : A\vec{x} = \vec{b} \text{ is consistent} \} \) is a subspace of \( \mathbb{R}^4 \).

c) Let \( W \) be as in part b). Show that \( \vec{b}_1 = 3\vec{e}_1 + 2\vec{e}_2 + \vec{e}_3 - \vec{e}_4 \in W \).

d) Let \( \vec{b}_1 \) be as in part c). Let \( \vec{x}_1 \) be one vector that satisfies \( A\vec{x}_1 = \vec{b}_1 \). Explain why the solution set of \( A\vec{x} = \vec{b}_1 \) equals the set of all vectors of the following form: \( \vec{x}_1 \) plus a solution of the equation in part a).
3. a) Let $A = \begin{bmatrix} 4 & -3 \\ -3 & -4 \end{bmatrix}$. Determine a diagonal matrix $D$ with real entries, and an orthogonal matrix $S$ for which $A = SDS^{-1}$. \textbf{Multiply out} $SDS^{-1}$ to check that your answer is correct.

b) Let $B = \begin{bmatrix} 4 & 3 \\ -3 & -4 \end{bmatrix}$. Determine whether $B$ is similar to the matrix $A$ (from part a) in $\mathbb{R}^{2 \times 2}$. 
4. Let $P_4$ be, as usual, the linear space of polynomials of degree $\leq 4$.
   
   a) Specify an isomorphism $\Phi : P_4 \to \mathbb{R}^n$ for some $n$. Explain why it is an isomorphism.

   b) True or false: Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Explain.

   c) True or false: There exists an inner product on $P_4$ with the property that $\{1, t, t^2, t^3, t^4\}$ is an orthonormal set. Explain.

   d) (Convince yourself that $\mathfrak{A} = \{2, 3 - t, 4t - t^2, 5 - t^3, 6 - t^4\}$ is a basis of $P_4$.) Determine the $\mathfrak{A}$-coordinate vector of $t^4 + t$.

5. a) Let $V$ be a linear space. Suppose that $\lambda$ is an eigenvalue of the linear transformation $T : V \to V$. Derive the fact that $\lambda^2$ is an eigenvalue of $T^2$.

   b) Determine all matrices in $\mathbb{R}^{3 \times 3}$ that are both symmetric and orthogonal, and describe them geometrically. [Suggestion: Express the two conditions in terms of “transpose.”]
6. Determine the matrix (for the standard basis of \( \mathbb{R}^5 \)) of the orthogonal projection of \( \mathbb{R}^5 \) onto the “plane” with equations \( x_1 - x_5 = 0, \ x_1 + x_2 + x_3 + x_4 = 0 \).

7. For which \( a \in \mathbb{R}, \ b \in \mathbb{R} \) does the matrix \( A = \begin{bmatrix} 2 & 0 & 0 \\ b & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \) have an eigenbasis (for \( \mathbb{R}^3 \))? When it does, specify an eigenbasis (depending on \( a \) and \( b \)).
8. Let $V$ be $\text{Span}\{1, \sin x, \cos x\}$. The dimension of $V$ is 3. We define a linear transformation $T : V \rightarrow \mathbb{R}^3$ by $T(f) = 2f(0)e_1 + f(\pi)e_2 - f(2\pi)e_3$.

a) Determine the rank of $T$.

b) Determine a basis for the kernel of $T$.

c) Let $D$ denote the linear operator on $V$ given by $D(f) = f'$. Determine the complex eigenvalues of $D$—that includes the real ones!—and the corresponding eigenspaces.

9. Let $A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$, where $b$ and $c$ are real scalars. Determine the set of values of $b$ and $c$ for which the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ is asymptotically stable (meaning: for all initial states, the state vector tends to $\vec{0}$ as $t \to \infty$).
[10] 10. Determine whether \( x_1^2 + 3x_1x_2 + 2x_2^2 = 1 \) is the equation of an ellipse.

[10] 11. a) Give an example of two \( 2 \times 2 \) real matrices that have the same characteristic polynomial yet they are not similar. \textbf{Explain.} \\

b) \textit{True or False:} If a matrix fails to diagonalize over \( \mathbb{R} \), it will diagonalize over \( \mathbb{C} \). \textbf{Explain.}