THE JOHNS HOPKINS UNIVERSITY
Krieger School of Arts and Sciences
FINAL EXAM - FALL 2005
110.201 – LINEAR ALGEBRA

Instructor: Professor Carel Faber
Duration: 180 minutes December 17, 2005

No calculators allowed **Total = 200 points**

NAME: Carel Faber

SECTION (weekday and time):

ETHICS PLEDGE:

I agree to complete this examination without unauthorized assistance from any person, materials, or device.

SIGNATURE: DATE:

1. [**30** points]

(a) [**25** points] Solve the following linear system:

$$\left|\begin{array}{ccccccccccc} & & & & x_3 & - & x_4 & - & x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & + & 2x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 3x_3 & + & 3x_4 & + & 3x_5 & = & 4 \\ 3x_1 & + & 6x_2 & + & 6x_3 & + & 3x_4 & + & 6x_5 & = & 6 \end{array}\right|.$$

$$\begin{pmatrix} 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 2 & 4 & 2 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 2 & 4 & 2 & 4 & 2 & 4 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 2 & 4 & 3 & 3 & 3 & 4 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 3 & 6 & 6 & 3 & 6 & 6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

~~the rref of the augmented matrix~~

$$\to \begin{pmatrix} 1 & 2 & 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$ the rref of the augmented matrix.

Thus x_1, x_3, x_5 are the leading variables, and x_2 and x_4 are the free variables. Put $x_2 = s$, $x_4 = t$.

Then: $\begin{cases} x_1 + 2x_2 + 3x_4 = 2 \\ x_3 - x_4 = 2 \\ x_5 = -2 \end{cases}$ So $\begin{cases} x_1 = 2 - 2s - 3t \\ x_3 = 2 + t \\ x_5 = -2 \end{cases}$

So $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 - 2s - 3t \\ s \\ 2 + t \\ t \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \\ -2 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

is the solution to the linear system.

(b) [**5** points] Check your answer.

$x_3 - x_4 - x_5 = 2 + t - t + 2 = 4$ ✓

$2x_1 + 4x_2 + 2x_3 + 4x_4 + 2x_5 = 2(2 - 2s - 3t) + 4s + 2(2+t)$
$+ 4t - 4 = 4 - 4s - 6t + 4s + 4 + 2t + 4t - 4 = 4$ ✓

$2x_1 + 4x_2 + 3x_3 + 3x_4 + 3x_5 = 2(2 - 2s - 3t) + 4s + 3(2+t)$
$+ 3t - 6 = 4 - 4s - 6t + 4s + 6 + 3t + 3t - 6 = 4$ ✓

$3x_1 + 6x_2 + 6x_3 + 3x_4 + 6x_5 = 3(2 - 2s - 3t) + 6s + 6(2+t)$
$+ 3t - 12 = 6 - 6s - 9t + 6s + 12 + 6t + 3t - 12 = 6$ ✓

2. [**30** points] Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & -3 & 2 & 5 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & -3 \end{bmatrix} \quad \text{and the vectors} \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -2 \end{bmatrix}.$$

(a) [**5** points] Show that $\vec{u}$ is contained in the image of A.

$$\vec{u} = A\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

(b) [**5** points] Show that $\vec{v}$ is contained in the kernel of A.

$$A\vec{v} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ -1 & -3 & 2 & 5 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & -3 \end{pmatrix}\begin{pmatrix} -1 \\ -1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -2-1+3 \\ 1+3+6-10 \\ -1+3-2 \\ -1-2-3+6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) [**10** points] Find a basis for the image of A such that $\vec{u}$ is one of the vectors in the basis. Show your work.

(i) Find rref(A), or ref(A):

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ -1 & -3 & 2 & 5 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & -3 & 2 & 5 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -3 & 3 & 6 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$= \text{ref}(A) = \text{rref}(A)$ here.

This says: (A has rank 2, thus $\dim(\text{im } A) = 2$, and) the first 2 columns of A form a basis of $\text{im}(A)$.

Both those columns are clearly not multiples of $\vec{u}$.

So $\left(\begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix} \right)$ [or $\left(\begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right)$]

or $\left(\begin{pmatrix} 1 \\ -3 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix} \right)$ are (for example) bases of the required form.

(d) [**10** points] Find a basis for the kernel of A such that $\vec{v}$ is one of the vectors in the basis. Show your work.

From (c): A has rank 2, size 4x4, thus $\text{nullity}(A) = \dim(\ker A) = 4 - 2 = 2$. To find a basis for $\ker(A)$:

$$\begin{cases} x_1 + x_3 + x_4 = 0 \\ x_2 - x_3 - 2x_4 = 0 \end{cases} : \quad \begin{matrix} x_3 = s & x_1 = -s-t \\ x_4 = t & x_2 = s + 2t \end{matrix} \quad \vec{x} = \begin{pmatrix} -s-t \\ s+2t \\ s \\ t \end{pmatrix}$$

(from rref(A))

$$\vec{x} = s \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\left(\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right)$ is a basis for $\ker(A)$.

Both vectors are clearly not multiples of $\vec{v}$ (note that $\vec{v} = 3 \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$).

So $\left(\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ -2 \end{pmatrix} \right)$ and $\left(\begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 3 \\ -2 \end{pmatrix} \right)$ are (for example) bases of the required form.

3. [**35** points] Consider the following vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}.$$

(a) [**5** points] Show that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal vectors.

$\vec{v}_1 \cdot \vec{v}_2 = 6 \cdot 2 + 3(-6) + 2 \cdot 3 = 12 - 18 + 6 = 0.$

(b) [**10** points] Show that $\vec{u}_1 = \frac{1}{7}\vec{v}_1$ and $\vec{u}_2 = \frac{1}{7}\vec{v}_2$ are unit vectors (i.e., vectors of length 1).

$\|\vec{v}_1\| = \sqrt{6^2+3^2+2^2} = \sqrt{36+9+4} = \sqrt{49} = 7$

and similarly $\|\vec{v}_2\| = 7.$

Thus $\|\vec{u}_1\| = |\frac{1}{7}| \; \|\vec{v}_1\| = \frac{1}{7} \cdot 7 = 1$

and similarly $\|\vec{u}_2\| = \frac{1}{7} \cdot 7 = 1.$

(c) [**10** points] Find a vector $\vec{u}_3$ such that $\mathcal{B} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is an orthonormal basis of $\mathbb{R}^3$.

Put $\vec{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, then it is easy to see that

$(\vec{u}_1, \vec{u}_2, \vec{x})$ is a basis of $\mathbb{R}^3$: $\det\begin{pmatrix}6 & 2 & 1\\ 3 & -6 & 0\\ 2 & 3 & 0\end{pmatrix} = \begin{vmatrix}3 & -6\\ 2 & 3\end{vmatrix} \neq 0$

Do Gram-Schmidt to $(\vec{u}_1, \vec{u}_2, \vec{x})$:

only $\vec{x}$ will be affected; (since $\vec{u}_1, \vec{u}_2$ already OK!)

$$\mathrm{proj}_{\langle \vec{u}_1, \vec{u}_2\rangle}(\vec{x}) = (\vec{u}_1 \cdot \vec{x})\vec{u}_1 + (\vec{u}_2 \cdot \vec{x})\vec{u}_2$$

$$= \frac{6}{7}\vec{u}_1 + \frac{2}{7}\vec{u}_2 = \begin{pmatrix}40\\6\\18\end{pmatrix} \cdot \frac{1}{49} \left(= \frac{1}{49}(6\vec{v}_1 + 2\vec{v}_2)\right)$$

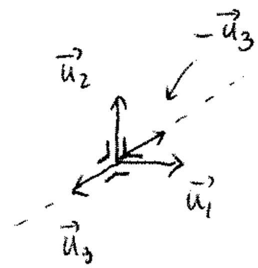
$$= \begin{pmatrix}40/49\\6/49\\18/49\end{pmatrix}.$$ Then $\vec{x}^{\perp} = \vec{x} - \mathrm{proj}(\vec{x}) = \begin{pmatrix}9/49\\-6/49\\-18/49\end{pmatrix} = \frac{3}{49}\begin{pmatrix}3\\-2\\-6\end{pmatrix}.$

Then $\vec{u}_3 = \dfrac{\vec{x}^{\perp}}{\|\vec{x}^{\perp}\|} = \dfrac{\begin{pmatrix}3\\-2\\-6\end{pmatrix}}{\left\|\begin{pmatrix}3\\-2\\-6\end{pmatrix}\right\|} = \dfrac{1}{7}\begin{pmatrix}3\\-2\\-6\end{pmatrix}.$

(Another correct answer: $-\frac{1}{7}\begin{pmatrix}3\\-2\\-6\end{pmatrix}$.)

(d) [**10** points] How many such vectors $\vec{u}_3$ are there? Motivate your answer. (Hint: think geometrically and/or draw a picture.)

There are two such vectors (the $\vec{u}_3$ you found, and $-\vec{u}_3$). In the basis $\mathcal{B}$, $\vec{u}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$; vectors orthogonal to $\vec{u}_1$ and $\vec{u}_2$ are $\begin{pmatrix}0\\0\\x\end{pmatrix}$; vectors of length 1 have $x=1$ or $x=-1$, so are $\vec{u}_3$ or $-\vec{u}_3$.



4. [**35** points]

(a) [**25** points] Fit a quadratic polynomial $kt^2 + \ell t + m$ to the data points $(-1, 6)$, $(0, -4)$, $(1, 2)$ and $(2, 4)$, using least squares. That is, find a least-squares solution to

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} k \\ \ell \\ m \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 2 \\ 4 \end{bmatrix}.$$

Put $A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} k \\ \ell \\ m \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 6 \\ -4 \\ 2 \\ 4 \end{pmatrix}$.

Instead of trying (in vain) to solve $A\vec{x} = \vec{b}$, we should solve $A^T A \vec{x} = A^T \vec{b}$.

$$A^T A = \begin{pmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 18 & 8 & 6 \\ 8 & 6 & 2 \\ 6 & 2 & 4 \end{pmatrix}.$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 24 \\ 4 \\ 8 \end{pmatrix}.$$

$$\left(\begin{array}{cccc} 18 & 8 & 6 & 24 \\ 8 & 6 & 2 & 4 \\ 6 & 2 & 4 & 8 \end{array}\right) \to \left|\begin{array}{cccc} 9 & 4 & 3 & 12 \\ 4 & 3 & 1 & 2 \\ 3 & 1 & 2 & 4 \end{array}\right| \xrightarrow{\mathrm{I}-2\cdot\mathrm{II}} \left|\begin{array}{cccc} 1 & -2 & 1 & 8 \\ 4 & 3 & 1 & 2 \\ 3 & 1 & 2 & 4 \end{array}\right|$$

$$\to \left|\begin{array}{cccc} 1 & -2 & 1 & 8 \\ 0 & 11 & -3 & -30 \\ 0 & 7 & -1 & -20 \end{array}\right| \xrightarrow{\mathrm{II}-\mathrm{III}} \left|\begin{array}{cccc} 1 & -2 & 1 & 8 \\ 0 & 4 & -2 & -10 \\ 0 & 7 & -1 & -20 \end{array}\right| \xrightarrow{\mathrm{III}-2\mathrm{II}} \left|\begin{array}{cccc} 1 & -2 & 1 & 8 \\ 0 & 4 & -2 & -10 \\ 0 & -1 & 3 & 0 \end{array}\right|$$

$$\to \left|\begin{array}{cccc} 1 & -2 & 1 & 8 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 10 & -10 \end{array}\right| \to \left|\begin{array}{cccc} 1 & -2 & 1 & 8 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right|.$$ OK: $m = -1$;

$\ell - 3m = 0$, $\ell = 3m = -3$; $k = 8 + 2\ell - m = 8 - 6 + 1 = 3$.

Solution: $3t^2 - 3t - 1$.

(b) [**10** points] Find the distance between the point $(6, -4, 2, 4)$ and the image of the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}.$$

$A\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}$ is the best we could do, it is the least-squares solution to $A\vec{x} = \vec{b} = \begin{pmatrix} 6 \\ -4 \\ 2 \\ 4 \end{pmatrix}$.

So $A\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -1 \\ 5 \end{pmatrix}$ is the closest vector in $\text{im}(A)$ to $\vec{b}$. (Note: $5, -1, -1, 5$ are the respective values of $3t^2 - 3t - 1$ in $t = -1, 0, 1, 2$!)

Then $\text{proj}_{\text{im}(A)}(\vec{b}) = \begin{pmatrix} 5 \\ -1 \\ -1 \\ 5 \end{pmatrix}$; $\vec{b}^{\perp} = \vec{b} - \text{proj}(\vec{b}) = \begin{pmatrix} 1 \\ -3 \\ 3 \\ -1 \end{pmatrix}$;

the distance $= \|\vec{b}^{\perp}\| = \sqrt{1^2 + 3^2 + 3^2 + 1^2} = \sqrt{20}$.

5. [**35** points] Let P_2 be the linear space of polynomials $f(t)$ of degree ≤ 2. Let T from P_2 to P_2 be the linear transformation given by

$$T(1) = 3 - t + t^2 \quad \text{and} \quad T(t) = -4 - t^2 \quad \text{and} \quad T(t^2) = -2 + 2t.$$

(a) [**5** points] Show that

$$T(1+t) = -(1+t) \quad \text{and that} \quad T(2+t^2) = 2(2+t^2).$$

$T(1+t) = T(1) + T(t) = 3 - t + t^2 - 4 - t^2 = -1 - t = -(1+t)$.

$T(2+t^2) = 2T(1) + T(t^2) = 6 - 2t + 2t^2 - 2 + 2t = 4 + 2t^2 = 2(2+t^2)$.

[Note: $1+t$ is an eigen"vector" with eigenvalue -1 and $2+t^2$ is an eigen"vector" with eigenvalue 2 for T.)

(b) [**10** points] Show that all solutions to

$$T(a + bt + ct^2) = 2(a + bt + ct^2)$$

are multiples of $2 + t^2$.

$T(a+bt+ct^2) = 3a - at + at^2 - 4b - bt^2 - 2c + 2ct$

$= (3a - 4b - 2c) + (-a - b + 2c)t + (a - b)t^2$

what we want $\rightarrow \stackrel{?}{=} \; 2a + 2b\,t + 2c\,t^2$.

Then: $\begin{cases} a - 4b - 2c = 0 \\ -a - 3b + 2c = 0 \\ a - b - 2c = 0 \end{cases}$ $\quad$ I − III: $b = 0$;

remaining equation: $a - 2c = 0$, or $a = 2c$.

So $a + bt + ct^2 = 2c + ct^2 = c(2+t^2)$, QED.

(c) [**10** points] Explain why the following statement is true: *If* the eigenvalue 2 of the linear transformation T has algebraic multiplicity two, then T is not diagonalizable.

We have just seen: the geometric multiplicity of the eigenvalue 2 of T equals one, since a basis for the eigenspace is $2+t^2$.
So, if the algebraic multiplicity of 2 were two, then we would not be able to find an eigenbasis for T, and so T would not be diagonalizable.

(d) [**10** points] *Prove* that the linear transformation T is not diagonalizable.

The matrix of T w.r.t. the standard basis $(1, t, t^2)$ is
$$A = \begin{pmatrix} 3 & -4 & -2 \\ -1 & 0 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

We know: $\lambda_1 = -1$, $\lambda_2 = 2$, are eigenvalues of A.
What is λ_3?
Easiest solution: $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace}(A) = 3+0+0 = 3$.
Thus: $\lambda_3 = 2$. Thus the algebraic multiplicity of the eigenvalue 2 is two; hence T is not diagonalizable.

6. [**35** points] You are sitting in the back of a lecture room and you are trying to read a 3×3 matrix written on the blackboard. The instructor's handwriting isn't always perfectly legible and you can read only the diagonal entries of the matrix A: $a_{11} = -2$, $a_{22} = 5$, and $a_{33} = 4$. You are told that 1 and 4 are eigenvalues of A.

(a) [**10** points] Determine the third eigenvalue of A and explain how you obtained your answer.

Like (5d): $\lambda_1 = 1$, $\lambda_2 = 4$;

$\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{trace}(A) = a_{11} + a_{22} + a_{33} = 7$.

So $\lambda_3 = 2$.

(b) [**10** points] Determine the determinant of A and explain how you obtained your answer.

$\det(A) = \lambda_1 \lambda_2 \lambda_3 = 1 \cdot 4 \cdot 2 = 8$.

(c) [**15** points] Determine the trace of A^2 and explain how you obtained your answer.

Recall: if $\vec{v}$ is an eigenvector of A with eigenvalue λ, then $A\vec{v} = \lambda\vec{v}$, hence $A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda(\lambda\vec{v}) = \lambda^2\vec{v}$, hence $\vec{v}$ is an eigenvector of A^2 with eigenvalue λ^2.

Now $1, 4, 2$ are all eigenvalues of A, and for each there exist eigenvectors (nonzero of course!!!). So $1^2, 4^2, 2^2$ are all three eigenvalues of A^2:

$1, 16$, and 4.

Then $\text{trace}(A^2) = 1 + 16 + 4 = 21$.