THE JOHNS HOPKINS UNIVERSITY
Krieger School of Arts and Sciences
FINAL EXAM - FALL 2005
110.201 – LINEAR ALGEBRA

Instructor: Professor Carel Faber
Duration: 180 minutes December 17, 2005

No calculators allowed **Total = 200 points**

NAME:

SECTION (weekday and time):

ETHICS PLEDGE:

I agree to complete this examination without unauthorized assistance from any person, materials, or device.

SIGNATURE: DATE:

1. [**30** points]

(a) [**25** points] Solve the following linear system:

$$\left|\begin{array}{ccccccccccc} & & & & x_3 & - & x_4 & - & x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 2x_3 & + & 4x_4 & + & 2x_5 & = & 4 \\ 2x_1 & + & 4x_2 & + & 3x_3 & + & 3x_4 & + & 3x_5 & = & 4 \\ 3x_1 & + & 6x_2 & + & 6x_3 & + & 3x_4 & + & 6x_5 & = & 6 \end{array}\right| .$$

(b) [**5** points] Check your answer.

2. [**30** points] Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & -3 & 2 & 5 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & -3 \end{bmatrix} \quad \text{and the vectors} \quad \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -2 \end{bmatrix}.$$

(a) [**5** points] Show that $\vec{u}$ is contained in the image of A.

(b) [**5** points] Show that $\vec{v}$ is contained in the kernel of A.

(c) [**10** points] Find a basis for the image of A such that $\vec{u}$ is one of the vectors in the basis. Show your work.

(d) [**10** points] Find a basis for the kernel of A such that $\vec{v}$ is one of the vectors in the basis. Show your work.

3. [**35** points] Consider the following vectors in $\mathbb{R}^3$:

$$\vec{v}_1 = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}.$$

(a) [**5** points] Show that $\vec{v}_1$ and $\vec{v}_2$ are orthogonal vectors.

(b) [**10** points] Show that $\vec{u}_1 = \frac{1}{7}\vec{v}_1$ and $\vec{u}_2 = \frac{1}{7}\vec{v}_2$ are unit vectors (i.e., vectors of length 1).

(c) [**10** points] Find a vector $\vec{u}_3$ such that $\mathcal{B} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is an orthonormal basis of $\mathbb{R}^3$.

(d) [**10** points] How many such vectors $\vec{u}_3$ are there? Motivate your answer. (Hint: think geometrically and/or draw a picture.)

4. [**35** points]

(a) [**25** points] Fit a quadratic polynomial $kt^2 + \ell t + m$ to the data points $(-1, 6)$, $(0, -4)$, $(1, 2)$ and $(2, 4)$, using least squares. That is, find a least-squares solution to

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} k \\ \ell \\ m \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 2 \\ 4 \end{bmatrix}.$$

(b) [**10** points] Find the distance between the point $(6, -4, 2, 4)$ and the image of the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}.$$

5. [**35** points] Let P_2 be the linear space of polynomials $f(t)$ of degree ≤ 2. Let T from P_2 to P_2 be the linear transformation given by

$$T(1) = 3 - t + t^2 \qquad \text{and} \qquad T(t) = -4 - t^2 \qquad \text{and} \qquad T(t^2) = -2 + 2t.$$

(a) [**5** points] Show that

$$T(1+t) = -(1+t) \qquad \text{and that} \qquad T(2+t^2) = 2(2+t^2).$$

(b) [**10** points] Show that all solutions to

$$T(a + bt + ct^2) = 2(a + bt + ct^2)$$

are multiples of $2 + t^2$.

(c) [**10** points] Explain why the following statement is true: *If* the eigenvalue 2 of the linear transformation T has algebraic multiplicity two, then T is not diagonalizable.

(d) [**10** points] *Prove* that the linear transformation T is not diagonalizable.

6. [**35** points] You are sitting in the back of a lecture room and you are trying to read a 3×3 matrix written on the blackboard. The instructor's handwriting isn't always perfectly legible and you can read only the diagonal entries of the matrix A: $a_{11} = -2$, $a_{22} = 5$, and $a_{33} = 4$. You are told that 1 and 4 are eigenvalues of A.

   (a) [**10** points] Determine the third eigenvalue of A and explain how you obtained your answer.

   (b) [**10** points] Determine the determinant of A and explain how you obtained your answer.

(c) [**15** points] Determine the trace of A^2 and explain how you obtained your answer.