LECTURES ON ZETA FUNCTIONS AND MOTIVES

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§0. Introduction:

These notes are based upon lectures given at Harvard University (Fall term 1991), Columbia University (Spring term 1992), Yale University (Whittemore Lectures, 1991), and MSRI.

Basically they represent propoganda for some beautiful recent ideas of Ch. Deninger and N. Kurokawa, shedding new light upon classical analogies between numbers and functions.

The central question we address can be provocatively put as follows: if numbers are similar to polynomials in one variable over a finite field, what is the analogue of polynomials in several variables? Or, in more geometric terms, does there exist a category in which one can define "absolute Descartes powers" Spec $\mathbb{Z} \times \cdots \times$ Spec $\mathbb{Z}$?

In [25], N. Kurokawa suggested that at least the zeta function of such an object can be defined via adding up zeroes of the Riemann zeta function. This agrees nicely with Ch. Deninger's representation of zeta functions as regularized infinite determinants [12]-[14] of certain "absolute Frobenius operators" acting upon a new cohomology theory.

In the first section we describe a highly speculative picture of analogies between arithmetics over $\mathbf{F}_q$ and over $\mathbb{Z}$, cast in the language reminiscent of Grothendieck's motives. We postulate the existence of a category with tensor product $\underset{\cdot}{\times}$ whose objects correspond not only to the divisors of the Hasse-Weil zeta functions of schemes over $\mathbb{Z}$, but also to Kurokawa's tensor divisors. This neatly leads to the introduction of an "absolute Tate motive" $\mathbb{T}$, whose zeta function is $\frac{s-1}{2\pi}$, and whose zeroth power is "the absolute point" which is the base for Kurokawa's direct products. We add some speculations about

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the role of **T** in the "algebraic geometry over a one-element field," and in clarifying the structure of the gamma factors at infinity.

The rest of the notes are devoted to more technical aspects of Kurokawa's tensor product. In the second section, we develop the classical Mellin transform approach to infinite determinants which is very convenient in dealing with tensor products. We slightly generalize the setting of [37] and [5] allowing logarithms in the asymptotic expansion of theta functions because they appear in our applications.

Finally, we discuss some examples of functions which were introduced independently of Kurokawa's construction but admit a natural tensor decomposition into simpler functions. They involve Barnes's multiple gamma functions, the Cohen-Lenstra zeta function, and a version of Wigner-Bloch-Zagier polylogarithms, studied independently by N. Kurokawa and M. Rovinskii.

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§1. Absolute Motives?

1.1 Comparing zeta functions of a curve over $\mathbf{F}_q$ and of $\mathbb{Z}$.

Let V be a smooth absolutely irreducible curve over $\mathbf{F}_q$. Its zeta function is

$$Z(V,s)=\sum_{\mathfrak{a}}\frac{1}{N(\mathfrak{a})^s}=\prod_{\mathfrak{p}}\frac{1}{1-N(\mathfrak{p})^{-s}} \tag{1.1}$$

where $\mathfrak{a}$ runs over $\mathbf{F}_q$-rational effective 0-cycles and $\mathfrak{p}$ runs over closed points of V. Denote by $V(\mathbf{F}_{q^f})$ the set of geometric points rational over $\mathbf{F}_{q^f}$ (for some fixed closure $\overline{\mathbf{F}}_q$). If we define $V(\mathbf{F}_{q^f})^o=\{x\in V(\mathbf{F}_{q^f})\mid \mathbf{F}_q(x)=\mathbf{F}_{q^f}\}$, we find from (1.1) that

$$Z(V,s)=\prod_{f=1}^{\infty}\left(\frac{1}{1-q^{-fs}}\right)^{\frac{\#V(\mathbf{F}_{q^f})^o}{f}} \tag{1.2}$$

One can easily calculate $\#V(\mathbf{F}_{q^f})^o$ via $\#V(\mathbf{F}_{q^d})$ with $d|f$; and this last function is given by a Lefschetz type formula

$$\#V(\mathbf{F}_{q^f})=\sum_{w=0}^{2}(-1)^w\mathrm{Tr}(Fr^f\mid H^w(V))=1-\sum_{j=0}^{2g}\phi_j^f+q^f, \tag{1.3}$$

where Fr is the Frobenius endomorphism acting on étale ℓ-adic cohomology groups of V, ϕ_j are its eigenvalues, and g is the genus of V. The Riemann conjecture proved by A. Weil states that ϕ_j are algebraic integers satisfying $|\phi_j|=q^{\frac{1}{2}}$. Combining (1.3) and (1.2) one gets a weight decomposition of $Z(V,s)$:

$$Z(V,s)=\prod_{w=0}^{2}Z(h^w(V);s)^{(-1)^{w-1}}=\frac{\prod_{j=1}^{2g}(1-\phi_jq^{-s})}{(1-q^{-s})(1-q^{1-s})}=$$

$$=\prod_{w=0}^{2}\det\Big((id-Fr\cdot q^{-s})\Big|H^w(V)\Big)^{(-1)^{w-1}}. \tag{1.4}$$

The weight w component is interpreted as the zeta function of "a piece" of V which is denoted $h^w(V)$ and is called "the pure weight w submotive of V." It is an entire function of s of order 1 (actually a rational function of q^{-s}), and the weight w is just the doubled real part of its zeroes.

Ch. Deninger in [12] suggested that a similar decomposition of the classical Riemann zeta function should be written as

$$Z(\overline{\mathrm{Spec}\,\mathbb{Z}},s):=2^{-\frac{1}{2}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=$$

$$=\frac{\prod_\rho\frac{s-\rho}{2\pi}}{\frac{s}{2\pi}\frac{s-1}{2\pi}}\overset{?}{=}\prod_{w=0}^{2}\mathrm{DET}\left(\frac{s\cdot id-\Phi}{2\pi}\mid H^w_?(\overline{\mathrm{Spec}\,\mathbb{Z}})\right)^{(-1)^{w-1}} \tag{1.5}$$

(see also a remark in [23], p. 335). There the notation $\prod_\rho$, as well as DET refers to the "zeta regularization" of infinite products. By definition,

$$\prod_{i\in I}\lambda_i:=\exp\left(-\frac{d}{dz}\sum_i\lambda_i^{-z}\Big|_{z=0}\right) \tag{1.6}$$

whenever the Dirichlet series in the r.h.s. of (1.6) converges in some half-plane and can be holomorphically extended to a neighborhood of zero (this involves a choice of arguments of λ_i).

The second equality in (1.5) is a theorem which Deninger deduces for $Re(s)\gg 0$ from a classical explicit formula. The last equality postulates the existence of a new cohomology theory $H_?$, endowed with a canonical "absolute Frobenius" endomorphism Φ. The Gamma-factor in (1.5), of course, should be interpreted as the Euler factor at infinity. Compactifying Spec $\mathbb{Z}$ to $\overline{\mathrm{Spec}\,\mathbb{Z}}$ then makes it similar to a projective curve over $\mathbf{F}_q$ rather than to an affine one. If Riemann's conjecture is true, the "absolute weights" ω of a factor (1.5) should again be the doubled real part of its zeroes.

The formal parallelism between (1.4) and (1.5) can be made even more striking.

Firstly, the factors in (1.4) can be written as infinite determinants as well: according to Deninger.

$$1 - \lambda q^{-s} = \prod_{\{\alpha | q^{\alpha} = \lambda\}} \frac{\log q}{2\pi i}(s - \alpha). \tag{1.7}$$

and the relevant cohomology spaces and Φ can be constructed in an elementary and functorial way from, say, étale cohomology ([13],[14]).

Secondly, the denominator in (1.4) is the inverse zeta function of $\mathbf{P}^1_{\mathbf{F}_q}$, or equivalently, the zeta function of the motive $\mathbf{L}^0 \oplus \mathbf{L}^1$ where $\mathbf{L}$ is the Tate motive over $\mathbf{F}_q$. In a similar fashion we suggest that the denominator in (1.5) should be looked upon as the inverse zeta function of an "absolute motive" $\mathbb{T}^0 \oplus \mathbb{T}^1$ where $\mathbb{T}$ is the absolute Tate motive (something like $\mathbf{L}$ over a "field of one element"). We will introduce $\mathbb{T}$, or rather its zeta function in §1.6 after reviewing briefly multidimensional schemes and Kurokawa's tensor product.

1.2 Zeta and motives over $\mathbf{F}_q$.

The parallelism discussed in §1.1 is expected to persist in higher dimensions. However, a very large part of the overall picture, especially the global one, remains conjectural. We will try to describe the relevant facts and hypotheses in the same format over $\mathbf{F}_q$ and $\mathbb{Z}$.

A. The definition and the weight decomposition of zeta.

Let $\mathrm{Var}/\mathbf{F}_q$ be the category of smooth projective varieties V defined over $\mathbf{F}_q$. For every V, one defines $Z(V,s)$ by the same formulas (1.1) as for curves. A. Weil conjectured, and A. Grothendieck with collaborators proved that

$$Z(V,s) = \prod_{w=0}^{2\dim V} \det\left((id - Fr \cdot q^{-s}) \,\middle|\, H^w(V)\right)^{(-1)^{w-1}}$$
$$= \prod_w Z(h^w(V), s)^{(-1)^{w-1}}. \tag{1.8}$$

P. Deligne proved the Riemann-Weil conjecture: $Re(\rho) = \frac{w}{2}$ for every real root ρ of $Z(h^w(V), s)$ (actually, in a considerably more general setting involving sheaf cohomology).

B. Various cohomology theories.

The formula (1.8) is essentially equivalent to a Lefschetz type formula counting the number of fixed points of (the powers of) the Frobenius endomorphism. The calculation itself is a formal consequence of several standard properties of a cohomology theory, including $H^{\cdot}(V \times W) = H^{\cdot}(V) \otimes H^{\cdot}(W)$, and $H^{\cdot}(\mathrm{Spec}\,\mathbf{F}_q) = E$ (coefficient field of cohomology theory). For details, see [23].

An important byproduct of the work of Grothendieck was a realization that there exists not one but many various cohomology theories with necessary properties, whose interrelations are otherwise not obvious. For example, the fact that the decomposition (1.8) constructed for $H^{\cdot} = H^{\cdot}_{\acute{e}t,\ell}$ with various $\ell \nmid q$ does not depend on ℓ, is not at all straightforward.

C. Motives: a universal cohomology theory?

Grothendieck, therefore, suggested that one look for a universal functor

$$h : (\mathrm{Var}/\mathbf{F}_q)^{\mathrm{opp}} \longrightarrow \mathrm{Mot}/\mathbf{F}_q$$

having (at least) the following properties:

The target category $\mathrm{Mot}/\mathbf{F}_q$ must be an additive E-linear (for a field of coefficients E) tensor category, with duality functor satisfying the standard axioms for finite dimensional linear spaces over E (technically speaking, a rigid tensor category). Furthermore, $\mathrm{Mot}/\mathbf{F}_q$ must be $\mathbb{Z}$-graded.

The functor h must satisfy the Künneth formula

$$h(V \times W) = h(V) \otimes h(W),$$

translate disjoint unions into direct sums, and verify additional axioms for the demonstration of Lefschetz' formula.

Every concrete cohomology theory like $H^{\cdot}_{\acute{e}t,\ell}$ must be a "realization" of the motivic cohomology, that is, must fit into a diagram of type

$$H^{\cdot}_{\acute{e}t,\ell} : \mathrm{Var}/\mathbf{F}_q \xrightarrow{h} \mathrm{Mot}/\mathbf{F}_q \xrightarrow{\text{étale } \ell\text{-adic realization}} \{\text{graded } \mathbf{Q}_\ell\text{-spaces}\}$$

A concrete proposal (developed in [25], [22]) for a construction of $\mathrm{Mot}/\mathbf{F}_q$ proceeds in three steps:

Step 1: For $V, W \in \mathrm{Var}/\mathbf{F}_q$, put

$$H(V,W) = C^d(V \times W), \qquad d = \dim W,$$

where $C^d(V \times W)$ is the space of d-codimensional cycles on $V \times W$ with coefficients modulo an adequate equivalence relation (numerical, algebraic, etc.) which we want to imply cohomological equivalence in our theories, e.g. numerical equivalence can be taken whenever we are interested only in Lefschetz formulas calculating intersection indices of algebraic cycles.

Introduce the multiplication of correspondences

$$C^{\cdot}(V \times W) \times C^{\cdot}(U \times V) \longrightarrow C^{\cdot}(U \times W)$$

in a classical way. This allows us to consider $C^{\cdot}(V \times W)$ as morphisms in a new category $\mathrm{Corr}/\mathbf{F}_q$, that of correspondences with coefficients in K. If $h(V)$ is the object V in $\mathrm{Corr}/\mathbf{F}_q$, put

$$h(V) \otimes h(W) = h(V \times W).$$

Step 2: Add formally to $\mathrm{Corr}/\mathbf{F}_q$, kernels and images of all projectors. In this way we get the category of effective motives $\mathrm{Mot}^+/\mathbf{F}_q$.

Step 3: In any $\mathrm{Corr}/\mathbf{F}_q$, we can prove that $\mathbf{P}^1 = \mathbb{I} \oplus \mathbf{L}_{\mathbf{F}_q}$ where $\mathbf{I} = h(\mathrm{Spec}\,\mathbf{F}_q)$ and $\mathbf{L}_{\mathbf{F}_q} = h^2(\mathbf{P}^1)$ in the sense that in any realization $\mathbf{L}_{\mathbf{F}_q}$ may have only weight two non-zero cohomology coinciding with that of $\mathbf{P}^1$.

$\mathbf{L}_{\mathbf{F}_q} = \mathbf{L}$ is called Tate's motive. Its version $\mathbf{L}_k$ can be defined over any ground field k. The functor $\bullet \mapsto \bullet \otimes \mathbf{L}_{\mathbf{F}_q}$ is the endomorphism of $\mathrm{Mot}^+/\mathbf{F}_q$ which is an autoequivalence. This allows us to adjoin formally negative powers of $\mathbf{L}$ and their tensor products by other effective motives. In this way, $\mathrm{Mot}^+/\mathbf{F}_q$ becomes enlarged to $\mathrm{Mot}/\mathbf{F}_q$, the category of pure motives.

One usually writes $M(n) = M \otimes \mathbf{L}^{\otimes(-n)}$. For motives over $\mathbf{F}_q$, one has two parallel decompositions:

$$h(\mathbf{P}^n) = \mathbf{I} \oplus \mathbf{L} \oplus \cdots \oplus \mathbf{L}^n;$$
$$\#\mathbf{P}^n(\mathbf{F}_q) = 1 + q + q^2 + \cdots + q^n,$$

so that one can naively imagine $\mathbf{L}^i$ as an avatar of an i-dimensional cell. For a curve $V/\mathbf{F}_q$, we have $h(V) = \mathbf{I} \oplus h^1(V) \oplus \mathbf{L}$.

The construction we have sketched can be performed over any base field k instead of $\mathbf{F}_q$. However, in order to prove all desirable properties of the category Mot/k one needs "standard conjectures" about algebraic cycles which remain unproved.

It became customary, therefore, to use the word "motive" loosely, referring to an object that has sufficiently many realizations in the cohomology theories: see [8], §0.12.

In the next section, we will speak about motives over $\mathbb{Q}$ in this vaguely defined sense, referring to [8] for more detailed statements. The point is that in the absence of a cohomology theory H; postulated by Deninger (cf. (1.5)), zeta functions remain our only observables, and all motives of [8] have well defined zetas.

D. Relation to zetas.

Let M be a motive, for simplicity, of pure weight w. This means that it admits a system of ℓ-adic realizations, $\ell \nmid q$. Assume that $\det((id - Fr \cdot q^{-s}) \mid H^w_{\text{ét},\ell}(M))$ is a polynomial of q^{-s} with integral coefficients independent of ℓ. This is, by definition, $Z(M,s)$.

The basic formulas relating motives to zetas are

$$Z(M \oplus N, s) = Z(M,s)Z(N,s),$$
$$Z(M \otimes N, s) = Z(M,s) \otimes Z(N,s)$$

where the r.h.s. tensor product of zeta functions means that Frobenius eigenvalues of the product are all pairwise products of Frobenius eigenvalues of factors (cf. §1.4) below. It follows that roots of $Z(M \otimes N, s)$ constitute a subfamily of the family of pairwise sums of roots of $Z(M,s)$ and $Z(N,s)$.

1.3 Zetas and motives over number fields.

A′. The definition and conjectural weight decomposition of zeta.

Let now V denote a smooth projective variety over a number field k. Serre [34]. and more generally Deligne [8], suggested the definition of the weight w factor of the zeta function of V:

$$\Lambda(h^w(V),s) := L_\infty(h^w(V),s) \prod_\nu L_\nu(h^w(V),s). \tag{1.13}$$

Here ν runs over finite places of k. The ν-Euler factor L_ν is defined by

$$(1.14)\qquad L_\nu(h^w(V),s) = \det\left((id - Fr_\nu \cdot N(\nu)^{-s}) \;\middle|\; H^w_{\text{ét}}(V)^{I_\nu}\right)^{-1}$$

where Fr_ν is a (geometric) Frobenius element lying in the decomposition subgroup $D_\nu \subset \mathrm{Gal}(\bar{k}/k)$, and $I_\nu \subset D_\nu$ is the inertia subgroup. Equation (1.14) is only well defined if the determinant has rational coefficients independent of ℓ, and this is true for almost all ν which are points of good reduction of an integral model of V.

The infinite Euler factor L_∞ is determined by the Hodge realization of V, corresponding to infinite places $\epsilon : k \hookrightarrow \mathbf{C}$. It is again the product of factors corresponding to all places. To describe such a factor, we can assume that V is defined over $\mathbf{R}$ or $\mathbf{C}$. Put

(1.15)

$$\Gamma_{\mathbf{R}}(s) = 2^{-\frac{1}{2}}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right); \quad \Gamma_{\mathbf{C}}(s) = (2\pi)^{-s}\Gamma(s);$$
$$h(p,q) = \dim H^{p,q}_{\mathbf{C}}(V);$$
$$h(p,\epsilon) = \dim\{x \in H^{p,p}_{\mathbf{C}}(V) \mid F_\infty(x) = (-1)^{p+\epsilon}x\}, \quad \epsilon = \pm 1$$

where $H^{p,q}_{\mathbf{C}}$ is the Hodge cohomology, and F_∞ is induced by the complex conjugation (Frobenius at infinity).

Then for a complex (resp. real) place σ of k we put

$$L_{\sigma,\mathbf{C}}(h^w(V),s) = \prod_{\substack{p<q\\ p+q=w}} \Gamma_{\mathbf{C}}(s-p)^{h(p,q)} \cdot \Gamma_{\mathbf{R}}\left(s-\frac{w}{2}\right)^{h(p,+)} \cdot \Gamma_{\mathbf{R}}\left(s-\frac{w}{2}+1\right)^{h(p,-)}$$

(for w odd, omit the last two factors),

$$L_{\sigma,\mathbf{R}}(h^w(V),s) = \prod_{p+q=w} \Gamma_{\mathbf{C}}(s-\min(p,q))^{h(p,q)}.$$

The analytic behavior of (1.13) is described by three basic conjectures:

A′1. $\Lambda(h^w(V),s)$ admits a meromorphic continuation to the whole comples plane, and satisfies the usual functional equation of the type $s \mapsto w+1-s$. (For a more precise form of this equation see [8] and [34]).

A′2. $\Lambda(h^w(V),s)$ may have poles only at $s = \frac{w+1}{2} \pm \frac{1}{2}$ for even w. The order of poles is (Tate's conjecture)

$$c(V,w) = \textit{rank of the subgroup } H^w(V) \textit{ generated by } k\textit{-rational algebraic cycles.}$$

A′3. The generalized Riemann conjecture:

$$\textit{roots of } \Lambda(h^w(V),s) \textit{ lie on } Re(s) = \frac{w+1}{2}.$$

Motivated by these conjectures and analogies with the finite characteristic case, we put:

$$\omega = w + 1 := \textit{absolute weight},$$

replace $h^w(V)$ by $\mathbf{H}^{w+1}(\bar{V})$, (piece of $\bar{V}$ of absolute weight $w+1$), and define its zeta function by

$$(1.16)\qquad Z(\mathbf{H}^{w+1}(\bar{V}),s) = \begin{cases} \Lambda(h^w(V),s)\cdot\left(\frac{(s-\frac{w}{2})(s-\frac{w+2}{2})}{4\pi^2}\right)^{c(V,w)}, & w \equiv 0(2);\\ \Lambda(h^w(V),s)\cdot\left(\frac{s-\frac{w+1}{2}}{2\pi}\right)^{c(V,w-1)+c(V,w+1)}, & w \equiv 1(2). \end{cases}$$

Conjecturally it is an entire function of order 1.

Deligne [8] defined $\Lambda(M,s)$ for more general motives over k. One can easily extend the definition of $Z(M,s)$ to them. There is one essential difference between zeta functions over $\mathbf{F}_q$ and k: in the global case, no analogue of the Dirichlet series representation for the alternating product of Λ's (as in (1.8)) is known. One might expect that such a representation should be connected with an Arakelov (arithmetically compactified) model $\bar{V}$ of V (this is why we put $\bar{V}$ in (1.16)). Such models possess a good deal of geometric properties, in particular, a group of 0-cycles and the degree map. Could it be that they lead to different type zetas? Already for Spec $\mathbf{Z}$ and Riemannian zeta this question is meaningful and unresolved.

B′. Various cohomology theories? We expect that (1.16) has a Deninger type representation (possibly up to an exponential factor)

$$(1.17)\qquad Z(\mathbf{H}^w(\bar{V}),s) = \prod_\rho \frac{s-\rho}{2\pi} = \mathrm{DET}\left(\frac{s\cdot id - \Phi}{2\pi} \;\middle|\; H^w_?(\bar{V})\right),$$

where $H_?^w(\bar{V})$ is an unknown cohomology theory with coefficients in $\mathbb{C}$, taking its values in infinite dimensional spaces in general, and DET is the zeta regularized infinite determinant, as in (1.6).

For some interesting suggestions about $H_?^w(\overline{\mathrm{Spec}\,\mathbb{Z}})$, see [38] and [16], compare also [2]. A conceptual basis for such a representation should be some kind of trace formula, rather than Lefschetz' formula: cf. [14].

By analogy with the $\mathbf{F}_q$ case, we expect different realizations for $\mathbf{H}^w(\bar{V})$. It is almost certain that ℓ-adic realizations are given by Iwasawa's construction, at least for spectra of rings of algebraic integers. If this is true, then the dependence on ℓ of the ℓ-adic cohomology seems to be much stronger than over $\mathbf{F}_q$. For example, the Leopoldt-Kubota zeta function has only a finite number of zeros (generally none) and their relation to archimedean zeroes is quite mysterious.

Summarizing, after works of Arakelov, Faltings, Bismut, Gillet, and Soulé, putting arithmetic geometry on a firm basis, we are now on a quest for an arithmetic topology.

C′. Absolute motives via correspondences? This approach seems totally out of reach at the moment because of the abscence of an absolute direct product of $\mathbb{Z}$-schemes. As a result, we have no idea about morphisms of absolute motives.

D′. Absolute motives via zetas. We introduce absolute motives as mythical animals corresponding to natural factors of zetas (as $H^w(\bar{V})$), and imagine operations on them imitating (1.12). However, since zetas have generally infinitely many zeroes, we must first discuss their tensor products in more detail.

1.4 Kurokawa's tensor product.

Let G be a connected one-dimensional algebraic group over an algebraically closed field k. This means that $G = G_a$, G_m or E, an elliptic curve. Let A be the ring of rational functions on a normal projective model of G whose divisors are supported by $G(k)$ (no restriction for $G = E$). Denote by U the group of non-vanishing rational functions whose divisors do not intersect $G(k)$: it is k^* for G_a and E, and $\{k^* t^m \mid m \in \mathbb{Z}\}$ for $G_m = \mathrm{Spec}\, k[t, t^{-1}]$. We will write $f \sim g$ for $f, g \in A$ if $f = ug$, $u \in U$. If $\mathrm{div}\, f = \sum_i m_i(x_i)$, $\mathrm{div}\, g = \sum_j n_j(y_j)$; $x_i, y_j \in G(k)$ we define $f \otimes g \in A$ up to a factor u by the condition

$$\mathrm{div}\,(f \otimes g) = \sum_{i,j} m_i n_j (x_i y_j),$$

where we write xy for the product of x and y in the sense of the group law in $G(k)$. In particular, for $G_a = \mathrm{Spec}\, k[t]$, and $G_m = \mathrm{Spec}\, k[t, t^{-1}]$ we have respectively

$$(1.19) \qquad G_a: \quad \prod_i (t - a_i)^{m_i} \otimes \prod_j (t - b_j)^{n_j} \sim \prod_{i,j} (t - a_i - b_j)^{m_i n_j}.$$

$$(1.20) \qquad G_m: \quad \prod_i (t - a_i)^{m_i} \otimes \prod_j (t - b_j)^{n_j} \sim \prod_{i,j} (t - a_i b_j)^{m_i n_j}$$

From (1.18) it follows easily that this tensor product is distributive with respect to the usual product

$$(1.21) \qquad fh \otimes g \sim (f \otimes g)(h \otimes g); \quad f \otimes gh \sim (f \otimes g)(f \otimes h).$$

Consider the category of pairs (H, Φ) where H is a finite dimensional vector space over k, and Φ is an endomorphism (resp. an automorphism) of H. Put $d_{(H,\Phi)}(t) = \det(t \cdot id_H - \Phi)$. Then the rules (1.19) (resp. (1.20)) reflect the following two possible definitions of the tensor product of two such pairs:

$$(1.22) \qquad G_a: \quad (H_1, \Phi_1) \otimes (H_2, \Phi_2) = (H_1 \otimes H_2,\ \Phi_1 \otimes id_{H_2} + id_{H_1} \otimes \Phi_{H_2})$$

$$(1.23) \qquad G_m: \quad (H_1, \Phi_1) \otimes (H_2, \Phi_2) = (H_1 \otimes H_2,\ \Phi_1 \otimes \Phi_2).$$

Of course (1.23) and (1.20) are the usual tensor product of étale cohomology spaces and Frobenius eigenvalues, while (1.22) and (1.19) are Kurokawa's suggestions for tensoring global zetas, except for complications connected with infinite dimensionality. Before discussing these complications let us mention that for $k = \mathbb{C}$ one can connect all three cases by defining using the coverings

$$(1.24) \qquad G_a(\mathbb{C}) \xrightarrow{\exp} G_m(\mathbb{C}) \xrightarrow{/(q^{\mathbb{Z}})} E(\mathbb{C}).$$

Ch. Deninger's construction of $H_?$ over $\mathbf{F}_q$ from the étale cohomology is a kind of delooping functor going backwards along the exp arrow in (1.24).

Let us now consider the $G = G_a$, $k = \mathbf{C}$ case, but extend A to a ring of meromorphic functions of finite growth with a possible exxential singularity at infinity. Such a function is defined by its divisor up to a factor $\exp(P(t))$, $P(t) \in \mathbb{C}[t]$, where the degree of P is bounded by the growth order. We will write $f \sim g$ for $f = \exp(P(t))g$.

Several difficulties may arise in defining $f \otimes g$.

(i) The right hand side of (1.18) may not be defined. e.g. because for some z, the equation $x_i y_j = z$ may have an infinite number of solutions. This happens, e.g. if we put $f(t) = g(t) = \zeta(t)$, since for every zero ρ of $\zeta(s)$, $1-\rho$ is also a zero. We will discuss Kurokawa's suggestion for redefining (1.18) in this case below.

Even if $\sum m_i n_j (x_i y_j)$ is defined, it may not be a divisor of a meromorphic function, due to its limit points of $\{x_i + y_j\}$. Neither of these difficulties arise if f (or g) has a finite divisor. We will use the remark below in the definition of Tate's motive. Another important case, that of "directed families" of zeroes will be treated in §2.

(ii) There are different ways to interpret the r.h.s. of (1.19) when (1.18) is defined. It can be defined via, say, a Weierstrass product regularization. The advantage of this prescription is that it leads to a meromorphic (or even entire) function of the whole plane. However, it may be difficult to understand the properties of $f \otimes g$ in terms of f and g, e.g. those which relate zeroes to primes and lead to explicit formulae.

The zeta realization (1.6) is better behaved in this respect, mainly because (for left directed families of zeroes) the Dirichlet series $\sum_{a \in \mathrm{Spec}} (s-a)^{-z}$ can be written as a Mellin transform of the theta function

$$\Theta_{H,\Phi}(t) = \mathrm{Tr}\,(e^{\Phi t}) = \sum_{a \in \mathrm{Spec}\,\Phi} e^{at},$$

and theta functions are simply multiplied under Kurokawa's additive tensor multiplication (1.22):

(1.25) $$\Theta_{(H_1,\Phi_1)\otimes(H_2,\Phi_2)}(t) = \Theta_{(H_1,\Phi_1)}(t)\Theta_{(H_2,\Phi_2)}(t).$$

1.5 Absolute motives: the rules of the game.

We will imagine that "natural factors" of zeta functions of $\mathbb{Z}$-schemes of finite type correspond to absolute motives $\mathbf{M}$ which can be reconstructed from the zeroes of $Z(\mathbf{M}, s)$ up to an (unspecified) isomorphism relation. Absolute motives can be added and (sometimes) multiplied, the respective composition law is denoted $\dot\times$. Finally, every absolute motive $\mathbf{M}$ has a dual motive $\mathbf{M}^t$. The rules are

(1.26) $$Z(\mathbf{M} + \mathbf{N}, s) = Z(\mathbf{M}, s)Z(\mathbf{N}, s),$$

(1.27) $$Z(\mathbf{M} \dot\times \mathbf{N}, s) \sim Z(\mathbf{M}, s) \otimes Z(\mathbf{N}, s),$$

(1.28) $$Z(\mathbf{M}^t, s) = Z(\mathbf{M}, -s).$$

If $Z(\mathbf{M}, s)$ is entire, with zeroes on $Re(s) = \omega/2$, $\mathbf{M}$ is called pure of absolute weight ω. In particular, every motive M must define an absolute motive $\mathbf{H}(M)$, with the same zeta function.

1.6 Absolute Tate's motive T.

We define it by

(1.29) $$Z(\mathbf{T}, s) = \frac{s-1}{2\pi},$$

and more generally

(1.30) $$Z(\mathbf{T}^{\dot\times n}, s) = \frac{s-n}{2\pi}$$

for all non-positive n.

1.6.1. Proposition: *Let M be a Grothendieck motive over $\mathbf{F}_q$ (as in §1.2) or a Deligne motive over $\mathbf{Q}$ (as in §1.3). Then*

(1.31) $$\mathbf{H}(M(n)) = \mathbf{H}(M) \dot\times \mathbf{T}^{(-n)}$$

as absolute motives. In particular, $\mathbf{H}(\mathbf{L}_{\mathbf{F}_q}) = \mathbf{H}(Spec\mathbf{F}_q) \dot\times \mathbf{T}$.

This follows from the formula

$$Z(M(n),s) = Z(M, n+s) \tag{1.32}$$

(cf. [8], the first line of p. 319), and from (1.27), (1.19). There is an elementary variation of (1.32). Let V be a scheme of finite type over $\mathbf{Z}$, and let $L(V,s)$ be its naive zeta function defined by the Dirichlet series $\sum \frac{1}{N(\mathfrak{a})^s}$ taken over effective cycles. Let $V[T]$ be the affine line over V. Then an easy argument shows that

$$L(V[T],s) = L(V,s-1) \sim L(V,s) \otimes Z(\mathbb{T},s).$$

Hence $\mathbf{T}$ can be imagined as a motive of a one-dimensional affine line over an absolute point, $\mathbf{T}^0 = \bullet = \operatorname{Spec} \mathbf{F}_1$. On the other hand, (1.5) corresponds to the decomposition of the arithmetic curve $\overline{\operatorname{Spec} \mathbf{Z}}$ similar to (1.5): $\mathbb{H}(\overline{\operatorname{Spec} \mathbf{Z}}) = \mathbf{T}^0 \oplus \mathbb{H}^1(\overline{\operatorname{Spec} \mathbf{Z}}) \oplus \mathbf{T}$.

It would be interesting to develop more systematically "algebraic geometry over $\mathbf{F}_1$." A classical insight is that

$$GL(n, \mathbf{F}_1) = S_n \quad \text{(symmetric group)},$$

and, more generally, $\mathbf{F}_1$-points of a classical linear group form its Weyl group.

We now suggest that zeta functions of $\mathbf{F}_1$-motives are some rational functions with integral and half-integral real part singularities; in particular, $Z(\cdot,s) = \frac{s}{2\pi}$. Pure absolute weight functions are g.c.d.'s of the respective classical ones, e.g. $s-1 = \text{g.c.d.}(1-p_1^{1-s},\ 1-p_2^{1-s}):\ p_1 \neq p_2$. Can this be given a sheaf theoretic interpretation?

Recently there was an upsurge of activity in the domain of quantum groups where the quantization parameter is traditionally denoted by q. Miraculously, for $q = p^n$, p a prime, quantum enveloping algebras are directly related to geometry over $\mathbf{F}_q$. Can this be used for a better understanding of the $q=1$ case?

1.7 Euler factors at infinity.

In [12], Deninger establishes for the basic factors infinite determinant decompostions

$$\Gamma_{\mathbf{R}}(s)^{-1} := [2^{-\frac{1}{2}}\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})]^{-s} = \prod_{n\geq 0} \frac{s+2n}{2\pi}, \tag{1.33}$$

$$\Gamma_{\mathbf{C}}(s)^{-1} := [(2\pi)^{-s}\Gamma(s)]^{-1} = \prod \frac{s+n}{2\pi} \tag{1.34}$$

and compares them with (1.7).

From our viewpoint, however, (1.33) and (1.34) are fundamentally different from (1.7) because they are not pure, and actually involve infinitely many negative weights. One way to look at it is to imagine, say (1.34) as the zeta function of the "dual infinite dimensional projective space over $\mathbf{F}_1$," with the motive $\oplus_{n=0}^{\infty} \mathbb{T}^{-n}$. This cannot be literally true, however, because we expect Spec $\mathbb{C}$ to be highly degenerate at its "closed point," so that the gamma factor reflects not the whole cohomology of this closed factor, but only its inertia invariant part as in (1.14).

A different insight is suggested by comparison with Selberg's zeta function. Physically, it describes the motion of a free particle in a space of constant negative curvature. Selberg's trace formula for compact Riemann surfaces says, roughly speaking, that certain quantum mechanical averages (sums over eigenvalues of the Laplacian) can be calculated classically (via sums over geodesics). However, the classical side of the sum should be complemented by terms, corresponding to the geodesic motion "in imaginary time" which is quantum mechanics corresponds to tunnelling. Analytically, these extra terms are due to the singularities of the gamma factor of Selberg's zeta which involves Barnes's double gamma (cf. some more details below in §3).

From the point of view of number theory, closed geodesics correspond to primes, whereas eigenvalues of the Laplacian correspond to the critical zeroes of the zeta function.

The "imaginary time motion" may be held responsible for the fact that zeroes of $\Gamma(s)^{-1}$ are purely real, whereas the zeroes of all non-archimedean Euler factors are pure imaginary.

1.8 Mixed absolute motives?

P. Deligne constructed a cohomology theory for arbitrary (not necessarily smooth or proper) complex algebraic varieties with values in mixed Hodge structures which are extensions of pure Hodge structures. He has also developed a language for speaking about "mixed motives." It is natural to expect that something analogous must happen in the world of absolute motives. In particular, (1.34) might correspond to a nontrivial multiple extension of Tate's motives, instead of the direct sum. This possibility is intriguing, because extensions of Tate's motives over a field k, according to Beilinson's conjectures, are described by the K-theory of k, at least up to torsion, and are closely related to polylogarithms and values of Dedekind's zetas at integer points. If a similar picture exists over $\mathbf{F}_1$, it may give rise to an interesting new (or reinterpretation of an old) chapter of combinatorics.

§2. Infinite determinants and Mellin's transform:

In this section, we will reproduce some basic classical material about zeta-regularization (see formula (1.6)). Our presentation is inspired by [12], [5] and [37]. We also allow the presence of logarithms in the asymptotics (2.17).

2.1 Infinite products.

Let λ be a complex number, $\lambda \neq 0$. In order to choose a branch of λ^{-z}, z complex, we may choose an argument of λ (defined up to $2\pi i\mathbb{Z}$):

(2.1) $$\lambda = |\lambda| e^{i\alpha}, \quad \lambda^{-z} := e^{-(\log|\lambda| + i\alpha)z}.$$

We then have

(2.2) $$\exp\left(-\frac{d}{dz}\lambda^{-z}\Big|_{z=0}\right) = \exp\left(\log|\lambda| + i\alpha\right) = \lambda$$

independently of $\alpha = \arg\lambda$. Therefore, for a finite family of non-zero λ_ν's and an arbitrary choice of their arguments we have

(2.3) $$\exp\left(-\frac{d}{dz}\sum_\nu \lambda_\nu^{-z}\Big|_{z=0}\right) = \prod_\nu \lambda_\nu.$$

Let now $\Lambda = (\lambda_\nu)$ be an infinite family of nonzero complex numbers, $\alpha_\nu = \arg\lambda_\nu$, a choice of their arguments. Assume that

a) $\sum_\nu \lambda_\nu^{-z}$ *converges for sufficiently large* $Re(z)$.

b) *The function* $\sum_\nu \lambda_\nu^{-z}$ *admits a meromorphic continuation and is regular in a neighborhood of* $z = 0$.

We then put (sometimes omitting α_ν):

(2.4) $$\prod \lambda_\nu := \prod(\lambda_\nu, \alpha_\nu) = \exp\left(-\frac{d}{dz}\sum_\nu \lambda_\nu^{-z}\Big|_{z=0}\right).$$

The following properties easily follow from (2.1) - (2.4):

2.1.1 The regularized product $\prod(\lambda_\nu, \alpha_\nu)$ does not change, if *a finite number* of α_ν's are chosen differently (however, it may change otherwise).

2.1.2 The product $\prod_{i=1}^n \prod_{\nu\in N_i}(\lambda_\nu, \alpha_\nu) = \prod_{\nu\in\cup N_i}(\lambda_\nu, \alpha_\nu)$ whenever N_i are pairwise disjoint and the l.h.s. is defined.

2.1.3 Put $\zeta_\Lambda(z) = \sum_\nu \lambda_\nu^{-z}$. Then for complex c, and $\alpha = \arg c$

(2.5) $$\prod(c\lambda_\nu, \alpha + \alpha\nu) = c^{\zeta_\Lambda(0)} \prod(\lambda_\nu, \alpha_\nu).$$

If $\{\lambda_\nu\}$ is the spectrum of an operator Φ in a space H, $\zeta_\lambda(0)$ can be called a regularized dimension of (H, Φ) and we define $\mathrm{DET}(s \cdot id - \Phi)$ as $\prod_\lambda(s - \lambda)$ (with some choice of arguments).

2.1.4 Examples:

i) We first apply (2.4) and (2.5) to the case $\lambda_\nu = \nu + s$, where $\nu = 0, 1, 2, \ldots$, $s \in \mathbb{C}$, and $-\frac{\pi}{2} < \alpha_\nu < \frac{\pi}{2}$. Then Hurwitz's zeta function

(2.6) $$\zeta(s, z) = \sum_{\nu=0}^{\infty} \frac{1}{(\nu + s)^z}$$

analytically continues to $z \in \mathbb{C}\setminus\{1\}$, with

(2.7) $$\zeta(s, 0) = \frac{1}{2} - s; \quad \frac{d}{dz}\zeta(s, z)\Big|_{z=0} = \log\Gamma(z) - \frac{1}{2}\log 2\pi.$$

From this one easily deduces Deninger's formulas (1.33), (1.34), and (1.7). Formulas (2.7), in turn, can be proved by Mellin's transform discussed below.

ii) Using Eisenstein's series instead of (2.6), Kurokawa similarly proves the following formula for $Im(\tau) > 0$, $q_\tau = e^{2\pi i\tau}$, $-\pi < \arg(s + m + n\tau) < \pi$,

(2.8) $$\prod_{m,n\in\mathbb{Z}} (s + m + n\tau) = (1 - q_s)\prod_{n=1}^{\infty}(1 - q_s q_\tau^n)(1 - q_s^{-1} q_\tau^n)$$

which may be considered as a refinement of Kronecker's limit formula.

We will now define a class of families closed with respect to tensor products and convenient for the Mellin transform.

2.2 Directed families.

In §2.1, instead of families ("sets with indices") Λ, one can consider subsets of complex numbers with multiplicities assigned, that is, functions $m^\Lambda : \mathbb{C} \to \mathbb{Z}_{\geq 0}$ (or $\mathbb{Z}$, or even $\mathbb{C}$), $\lambda \mapsto m_\lambda$, whose support is discrete in $\mathbb{C}$.

2.2.1 Definition:

a) *Λ is left directed if*

$$\forall r \in \mathbf{R}, \quad \#\{\lambda \in \Lambda \mid Re(\lambda) > r\} < \infty, \tag{2.9}$$

$$\exists \beta > 0, \ \text{s.t.} \sum_{Re(\lambda) \geq -H} |m_\lambda| = O(H^\beta) \ \text{as} \ H \longrightarrow \infty. \tag{2.10}$$

b) *The theta function of a left directed family is*

$$\theta(t) = \theta_\Lambda(t) = \sum_{\lambda \in \Lambda} m_\lambda e^{\lambda t}, \tag{2.11}$$

and the theta function with parameter s is

$$\theta_{\Lambda,s}(t) = \theta_{\Lambda - s}(t) = \sum_{\lambda \in \Lambda} e^{(\lambda - s)t}. \tag{2.12}$$

c) *The tensor product of two families Λ_1, Λ_2 is defined if*

$$\forall \lambda \in \mathbb{C}, \quad \#\{\lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2 \mid \lambda_1 + \lambda_2 = \lambda\} < \infty, \tag{2.13}$$

and is given by $\Lambda = \{\lambda_1 + \lambda_2 \mid \lambda_i \in \Lambda_i\}$ and

$$m_\lambda^\Lambda = \sum_{\lambda_1 + \lambda_2 = \lambda} m_{\lambda_1}^{\Lambda_1} m_{\lambda_2}^{\Lambda_2}. \tag{2.14}$$

The following statements are straightforward.

2.22 Proposition:

a) *If Λ is left directed then $\theta_\Lambda(t)$ absolutely converges for every $t \in (0, \infty)$.*

b) *If, in addition, $Re(\lambda) < 0$ for all $\ell \in \Lambda$, then $\theta_\Lambda(t) = O(e^{-at})$ as $t \to +\infty$ for some $a > 0$.*

2.2.3 Proposition: *If Λ_1, Λ_2 are left directed, then $\Lambda_1 \otimes \Lambda_2$ is defined and left directed.*

2.24 Remarks and examples.

a) Every finite family is left directed.

b) $\mathbb{Z}_{\leq 0}$ with multiplicities 1 is left directed. Its n-th tensor power contains $-k$ with multiplicity $\binom{n+k-1}{n-1}$.

c) The tensor product (2.13), (2.14) is associative and commutative. The family $\{0, \textit{with multiplicity } 1\}$ is the identity for $\otimes$.

2.3 Mellin transform formulae.

Let Λ be left directed. Then in the region $(s, z) \in \mathbb{C}^2$, $Re(z) > \beta$ (see (2.10)), $Re(s) > \max_{\lambda \in \Lambda} Re(\lambda)$, we have

$$\zeta_\Lambda(s, z) := \sum_\lambda (s - \lambda)^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty \theta_{\Lambda,s}(t) t^{z-1} \, dt \tag{2.15}$$

where for $(s - \lambda)^{-z}$ we choose the determination which is real for $s - \lambda > 0$, z real. The r.h.s. series converges absolutely in this domain.

We want to use the r.h.s. of (2.15) in order to analytically continue the l.h.s. We first fix s with $Re(s) \gg 0$ and analytically continue in z. Put generally, for a continuous function $\theta(t)$, $t \in (0, \infty)$:

$$(\mathcal{M}_t \theta)(z) = \int_0^\infty \theta(t) t^{z-1} \, dt. \tag{2.16}$$

Assuming that $\theta(t) = O(t^\alpha)$ for $t \to +0$ and $O(t^\beta)$ for $t \to +\infty$, we see that (2.16) converges at 0 for $Re(z) > -\alpha$, and at ∞ for $Re(z) < -\beta$. Hence, if $\alpha < -\beta$, $(\mathcal{M}_t \theta)$ is defined and holomorphic in the strip $Re(z) \in (-\alpha, -\beta)$, and bounded in any strip $Re(z) \in (-\alpha + \epsilon, -\beta - \epsilon)$, $\epsilon > 0$.

More generally:

2.4 Proposition: *Let $\{i_a \mid a \geq 0\}$ be a sequence of complex numbers with $Re(i_a) \to +\infty$, $Re(i_a) \leq Re(i_{a+1})$, and $P_a(\log t)$ a sequence of polynomials in $\log t$. Assume that $\theta(t)$ admits an asymptotic expansion*

$$\theta(t) \sim \sum_{a=0}^\infty t^{i_a} P_a(\log t) \quad \text{as} \quad t \to +0, \tag{2.17}$$

and

(2.18) $$\theta(t) = O(t^{\beta}), \quad \beta < \alpha = Re(i_0) \quad \text{as} \quad t \to +\infty.$$

Then $(\mathcal{M}_t\theta)$ can be meromorphically extended to the half-plane $Re(z) < -\beta$, where its only singularities are poles at $z = -i_a$, with principal parts

(2.19) $$P_a\left(\frac{\partial}{\partial z}\right)\frac{1}{z+i_a}.$$

Proof: For every $m \geq 0$ such that $Re(i_{m+1}) > Re(i_m)$, we have

$$\theta(t) = \sum_{a=0}^{m} t^{i_a} P_a(\log t) + \Psi(t),$$

$$\Psi(t) = O(t^{i_{m+1}-\epsilon}) \quad \text{as} \quad t \to +0, \quad \text{for any } \epsilon > 0.$$

Hence $(\mathcal{M}_t\theta)$ is holomorphic in $Re(z) \in (-Re\, i_{m+1}, -\beta)$. On the other hand, the term $t^{i_a} P_a(\log t)t^{z-1}$ introduces the singularity

$$\int_0^1 t^{i_a} P_a(\log t)t^{z-1}\, dt$$

which is (2.19) up to a holomorphic function.

2.5 Analytic continuation to the right.

Replacing t by $1/t$ in the r.h.s. of (2.16) we find

$$(\mathcal{M}_t\theta)(-z) = -\int_0^{\infty} \theta(t^{-1})t^{z-1}\, dt.$$

Hence we can apply Proposition 2.4 if θ admits a similar asymptotic series at infinity.

When $\theta(t)$ is the theta function of a left directed family, the argument is slightly different. According to Proposition 2.2.2b,

$$\theta_\Lambda(t) := \sum m_\lambda e^{\lambda t} = \sum_{Re(\lambda)\geq 0} m_\lambda e^{\lambda t} + O\left(e^{-\epsilon t}\right) \quad \text{as} \quad t \to +\infty$$

for some $\epsilon > 0$, so that

$$(\mathcal{M}_t\theta_\Lambda(t))(z) = \Gamma(z) \sum_{Re(\lambda)\geq 0} m_\lambda(-\lambda)^{-z} + \phi(z)$$

where the sum is finite and $\phi(z)$ is holomorphic for $Re(z) > 0$. We use the formula

$$\int_0^{\infty} e^{\lambda t}t^{z-1}\, dt = \Gamma(z)(-\lambda)^{-z},$$

where the integral converges for $Re(\lambda) < 0$, $Re(z) > 0$, and then must be analytically continued in such a way that $(-\lambda)^{-z} = e^{-z\log(-\lambda)}$ with real $\log(-\lambda)$ for $Re(\lambda) < 0$.

2.6 Introducing the parameter s.

We will now apply Proposition 2.4 to the function (2.12) and its Mellin transform (2.15). If $\theta(t)$ satisfies (2.17) and (2.18) then $\theta_s(t) = e^{-st}\theta(t)$ satisfies the similar conditions:

(2.20) $$\theta_s(t) \sim \sum_{b=0}^{\infty} t^{j_b} Q_b(\log t; s) \quad \text{as } t \to +0,$$

(2.21) $$\{j_b\} = \{i_a + m \,|\, a \geq 0,\ m \geq 0\},$$

(2.22) $$Q_b(\log t; s) = \sum_{i_a+m=j_b} P_a(\log t)\frac{s^m}{m!}(-1)^m$$

because we can multiply (2.17) by $e^{-st} = \sum_{m=0}^{\infty}(-1)^m \frac{s^m}{m!}t^m$,

(2.23) $$\theta_s(t) = O(t^{\beta}) \quad \text{for} \quad Re(s) \geq 0 \quad \text{as } t \to +\infty.$$

We can now state the main result of this section.

2.7 Theorem: *Let Λ be a left directed family whose theta function admits an asymptotic expansion (2.17). Then its zeta function $\zeta_\Lambda(s,z)$ (2.15) admits a meromorphic continuation to the domain of all z and all s outside of horizontal left cuts $\{Re(s) \leq Re(\lambda),\ Im(s) = Im(\lambda) \mid \lambda \in \Lambda\}$. Furthermore:*

i) *For a given $n \geq 0$, $\zeta_\Lambda(s,z)$ is regular in a neighborhood of $z = -n$, if and only if for all $a \geq 0$, $m \geq 0$ with $i_a + m = n$ we have*

(2.24) $$P_a(\log t) = p_a \quad (\textit{a constant}).$$

ii) *If this condition is satisfied then*

(2.25) $$\zeta_\Lambda(s,-n) = (-1)^n n! \sum_{i_a+m=n} \frac{(-s)^m}{m!} p_a$$

iii) *In particular, $\zeta_\Lambda(s,z)$ is regular near $z=0$, if $P_a(\log t)$ is a constant for all $i_a = 0,-1,-2,\dots$, and then*

(2.26) $$\zeta_\Lambda(s,0) = \sum_{i_a \in \{0,-1,-2,\dots\}} (-1)^{i_a} p_a \frac{s^{|i_a|}}{|i_a|!}.$$

Proof: From (2.15), (2.19), and (2.22) one sees that $\zeta_\Lambda(s,z)$ near $z=n$ behaves as

$$\frac{1}{\Gamma(z)} \sum_{\substack{a\geq 0, m\geq 0\\ i_a+m=n}} \frac{(-s)^m}{m!} P_a\left(\frac{\partial}{\partial z}\right)\frac{1}{z+n} \quad + \quad O(z-n),$$

because $\Gamma(z)$ has a pole of the first order at $z=n$. Obviously, (2.24) - (2.26) follow from this.

As an easy application, we can now check the first formula (2.7). Hurwitz's zeta corresponds to

$$\theta(t) = \sum_{v=0}^{\infty} e^{-vt} = \frac{1}{1-e^{-t}} = \frac{1}{t} + \frac{1}{2} + O(t), \qquad t \mapsto +0,$$

so that $i_0 = -1$, $p_0 = 1$; $i_1 = 0, p_1 = \frac{1}{2}$, and $\zeta_\Lambda(s,0) = -s + \frac{1}{2}$ in view of (2.26).

The second formula (2.7) admits the following generalization (with a slight weakening).

2.8 Theorem: ([5]).*In the situation of Theorem 2.7, make the following additional assumptions:*

a) *The multiplicities m_λ of the left directed family Λ are non-negative integers.*

b) *In the asymptotic series (2.17), all i_a are integers and all P_a are constants p_a.*

Then

$$D(s) := \prod_{\lambda\in\Lambda} (s-\lambda)^{m_\lambda} := \exp\left(-\frac{d}{dz}\zeta(s,z)\Big|_{z=0}\right)$$

is an entire function of finite order with zeroes $\lambda \in \Lambda$ of multiplicity m_λ, whose logarithm in the cut s-plane admits the following asymptotic series as $Re(s) \to +\infty$.

(2.27) $$\log D(s) = -\frac{d}{dz}\zeta(s,z)\Big|_{z=0} \sim \sum_{a\geq 0} (-1)^{i_a} p_a \left(\frac{d}{ds}\right)^{i_a} (\log s)$$

where for negative m, $\left(\frac{d}{ds}\right)^m (\log s)$ should be interpreted as the unique m-th primitive of $\log s$ of the form $s^{|m|}(a_m \log s + b_m)$, that is

$$\left(\log s - \left(1 + \frac{1}{2} + \cdots + \frac{1}{|m|}\right)\right)\frac{s^{|m|}}{|m|!}.$$

(Notice that (2.26) can be rewritten similarly with 1 instead of $\log s$).

Sketch of proof:

A. One verifies that $D(s)$ is an entire function of finite growth, with zeroes $\lambda \in \Lambda$ of multiplicity m_λ. In fact, since m_λ are integers, one can check using (2.15) that $\log D(s)$ jumps by integral multiples of $2\pi i$ when s crosses a cut; the singularities of $\log D(s)$ in s are controlled by finite partial sums of $\frac{d}{dz}\left(\sum_\lambda m_\lambda (s-\lambda)^{-z}\right)\Big|_{z=0}$; and the growth order can be estimated by using the integral representation (2.15).

B. Therefore, $D(s)$ admits a Weierstrass-Hadamard representation

(2.28)
$$D(s) = \exp(P(s)) \prod \left(1 - \frac{s}{\lambda}\right)^{m_\lambda} \exp\left(m_\lambda\left(\frac{s}{\lambda} + \frac{1}{2}\frac{s^2}{\lambda^2} + \cdots + \frac{1}{N}\frac{s^N}{\lambda^N}\right)\right),$$

$p(s) \in \mathbf{C}[s], \qquad \deg(P(s)) \leq N.$

Taking logarithms and differentiating $r > N$ times, we get from (2.28) and (2.15):

(2.29)
$$\frac{(-1)^{r-1}}{(r-1)!}\left(\frac{d}{ds}\right)^r \log D(s) = \sum_{\lambda\in\Lambda} m_\lambda (s-\lambda)^{-r} = \frac{1}{(r-1)!}\int_0^\infty \theta_{\Lambda,s}(t)t^{r-1}\,dt.$$

From (2.17) one can deduce the asymptotic series for the Laplace transform (2.29) for large $Re(s)$. This asymptotic series can be obtained by applying $(-1)^{r-1}\left(\frac{d}{ds}\right)^r$ formally to the r.h.s. of (2.27).

C. It remains to show that an appropriate r-tuple integration of the resulting formula leads to (2.27). For more details, see [5], pp. 21-30.

2.9 Comparison with Hadamard-Weierstrass products.

For every left directed family Λ with non-negative integral multiplicities m_λ, one can construct an entire function having Λ as its divisor by forming a product (2.28). It is defined up to $\exp(P(s))$ with a polynomial $P(s)$.

Nevertheless, $D(s) := \prod_{\lambda\in\Lambda}(s-\lambda)^{m_\lambda}$ may not exist, due to the non-regularity of $\zeta_\Lambda(s,z)$ at $z=0$. One relevant case is $\Lambda = \{-p \mid p \text{ primes}\}$, with multiplicities one. In fact, it is known that $Re(z) = 0$ is a natural boundary for $\sum_p \frac{1}{(p+s)^z}$.

On the other hand, when $D(s)$ does exist, it can be uniquely defined in the class of entire functions of finite growth by the condition that $\log D(s)$ admits an asymptotic expansion (2.27) with unspecified constants p_a. In fact, if there are two functions $D_{1,2}(s)$, then $\log D_1(s) - \log D_2(s)$ must be a polynomial, but a non-vanishing polynomial does not admit an expansion (2.27).

2.10 Stability with respect to tensor products.

We recall that Kurokawa's tensor product corresponds to the product of theta functions, and if we stick to the zeta regularized determinants, we get well defined prescriptions

(2.30)
$$\prod_{\lambda\in\Lambda}(s-\lambda)^{m_\lambda} \otimes \prod_{\mu\in M}(s-\mu)^{n_\mu} := \prod (s-\lambda-\mu)^{m_\lambda+n_\mu}$$

$$\prod_{\lambda\in\Lambda}(s-\lambda)^{m_\lambda} = \exp\left(-\frac{d}{dz}\sum\frac{m_\lambda}{(s-\lambda)^z}\Big|_{z=0}\right)$$

(2.31)
$$= \exp\left(-\frac{d}{dz}\left(\frac{1}{\Gamma(z)}\int_0^\infty \sum_\lambda m_\lambda e^{(s-\lambda)t}t^{z-1}\,dt\right)\Big|_{z=0}\right).$$

analytic continuation in z being implied.

If expansions (2.17) of $\theta_\Lambda(t)$ and $\theta_M(t)$ have no logarithms, the same is true for $(\theta_\Lambda\theta_M)(t)$, so that tensor products also exist.

2.11 Standard regularization.

In (1.5), we quoted the formula due to Ch. Deninger and C. Soulé

$$\prod_\rho \frac{s-\rho}{2\pi} = \frac{s(s-1)}{4\pi^2}2^{-1/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

where ρ runs over all critical zeros of the Riemann $\zeta(s)$. In order to define corrected tensor powers of $\zeta(s)$, Kurokawa suggests the use of entire functions whose zeroes constitute only half the zeroes of zeta: $\{\rho \mid Im(\rho) > 0\}$ or $\{\rho \mid Im(\rho) < 0\}$, in this way $i\rho$ (respectively, $-i\rho$) become left directed. However, the separated products

$$\prod_{\substack{Im(\rho)>0 \\ <0}} \frac{s-i\rho}{2\pi}$$

do not exist because $\sum_{Im(\rho)>0} e^{i\rho t}$ admits an asymptotic expansion of the form (2.17), starting with terms $a\log t/t + b\log t$. We will discuss this in more detail below in §5. Here we only notice that singularities of this kind can be disposed of by subtracting from the theta function involved certain polynomials of some standard theta functions (e.g. starting with terms $t^{-1}, t^{-1}\log t$, and $\log t$).

As one such set we can choose

(2.32)

$$\theta_1(t) := \sum_{m,n\geq 1} e^{-mnt} \sim \frac{1}{t}(\log\frac{1}{t}+\gamma) + \frac{1}{4} - \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}t^{2n-1},$$

$$\theta_2(t) := \sum_{n\geq 1} e^{-nt} = \frac{e^{-t}}{1-e^{-t}} \sim \frac{1}{t} + O(1),$$

$$\theta_3(t) = \theta_1(t)/\theta_2(t).$$

(I am thankful for D. Zagier for (2.32)).

§3. Multiple gamma functions and Selberg's zeta function:

3.1 Gamma-divisors.

Put $\Lambda_0 = \{0, \text{ multiplicity } 1\}$ and for $n \geq 1$:

$$\Lambda_n = \{-k, \text{ multiplicity} \binom{n+k-1}{n-1} \mid k = 0,1,2,\dots\}.$$

We have $\Lambda_n = \Lambda_1^{\otimes n}$ (Kurokawa's tensor product corresponding to G_a).

3.2 Theta-functions and zeta functions.

From (2.11) and (2.14) we find:

(3.1)
$$\theta_{\Lambda_n}(t) = \theta_{\Lambda_1}(t)^n = \left(\frac{1}{1-e^{-t}}\right)^n := \sum_{i=-n}^{\infty} t^i\beta_i^{(n)}, \quad t \to +0$$

where $\{\beta_i^{(n)}\}$ are "n-th Bernoulli numbers," up to a normalization.

Obviously, Λ_n are left directed families, to which Theorems 2.7 and 2.8 are applicable. We have by definition

$$\zeta_{\Lambda_n}(s;z) = \sum_{k=0}^{\infty}\binom{n+k-1}{n-1}\frac{1}{(s+k)^z}.$$

This function has a meromorphic continuation to the entire z-plane, and in particular

(3.2)
$$\zeta_{\Lambda_n}(s;0) = \sum_{j=-n}^{0}\beta_j^{(n)}\frac{(-s)^{|j|}}{|j|!},$$

(3.3)
$$-\frac{d}{dz}\zeta_{\Lambda_n}(s;z)\Big|_{z=0} \sim \sum_{j=-n}^{\infty}(-1)^j\beta_j^{(n)}\left(\frac{d}{ds}\right)^j(\log s), \quad s \to +\infty.$$

3.3 Multiple gammas as infinite determinants.

It follows that the following expressions define an entire function of order $n \geq 1$ with divisor Λ_n :

(3.4)
$$\Delta_n(s) = \prod_{k=0}^{\infty}(s+k)^{\binom{n+k-1}{n-1}} = \prod_{k_1,\dots,k_n=0}^{\infty}(s+k_1+\dots+k_n)$$

We also put $\Delta_0(s) = s$. Define a family of real polynomials $P_n(s)$ by

(3.5)
$$P_0 \equiv 0; \quad P_n(1) = (-1)^{n-1}\log\Delta_n(1);$$
$$P_n(s+1) - P_n(s) = P_{n-1}(s), \quad n \geq 1.$$

Finally, introduce multiple gamma functions:

(3.6)
$$\Gamma_n(s) := \exp\Big(P_n(s)\Big)\Delta_n(s)^{(-1)^n}.$$

The following list of properties generalizes the classical $n = 1$ case.

3.3.1 Functional equations and higher factorials. The set of functions $\{\Gamma_n(s) \mid n \geq 0\}$ is the unique system of functions satisfying the following properties:

a) $\Gamma_n(s)^{(-1)^n}$ is an entire function of order n with divisor Λ_n; $\Gamma_0(s) = s$; $\Gamma_n(1) = 1$ for all n.

b) $\Gamma_n(s+1) = \Gamma_{n-1}(s)\Gamma_n(s), \quad n \geq 1$.

c) $\frac{d^{n+1}}{dx^{n+1}}\log\Gamma_n(x) \geq 0$ for all real $x \geq 1$.

Proof: First, a) follows from (3.6). To prove b), start with the identity

$$\binom{n+k-1}{n-1} = -\binom{n+k-1}{n-2} + \binom{n+k}{n-1}$$

and interpret it as

$$\{\text{order of } \Delta_n(s+1) \text{ at } s=-k-1\} = \tag{3.7}$$
$$-\{\text{order of } \Delta_{n-1}(s) \text{ at } s=-k-1\} + \{\text{order of } \Delta_n(s) \text{ at } s=-k-1\}$$

Using in addition the formula $D_{\Lambda-s_0}(s) = D_\Lambda(s+s_0)$ for $D_\Lambda(s) = \prod_{\lambda\in\Lambda}(s-\lambda)^{m(\lambda)}$ satisfying the Cartier-Voros condition (2.27), we get from (3.6)

$$\Delta_n(s+1) = \Delta_{n-1}(s)^{-1}\Delta_n(s). \tag{3.8}$$

Now, in Vignéras [36] it is proved that there exists a unique system of functions satisfying a), b), c). It is denoted $\{G_n(s)\}$ there. Since divisors of $G_n(s)$ and $\Delta_n(s)^{(-1)^n}$ coincide, we have $G_n(s) = \exp(P_n(s))\Delta_n(s)^{(-1)^n}$ where $P_n(s)$ is a polynomial. It has real coefficients because G_n and Δ_n are real on $\mathbf{R}_{>0}$.

From $G_n(1) = 1$ it follows that $P_n(1) = (-1)^{n-1}\log\Delta_n(1)$. Finally, the last functional equation in (3.5) is obtained by comparing the identity $G_n(s+1) = G_{n-1}(s)G_n(s)$ with (3.8).

The value of Γ_n at integers > 0 are "higher factorials:"

$$\Gamma_2(s+1) = 1!\cdots s!\,; \quad \Gamma_n(s+1) = \prod_{j=1}^{s}\Gamma_{n-1}(j). \tag{3.9}$$

One can use this remark in order to "calculate" the regularized infinite products in terms of asymptotic behaviour of (3.9) as $s \to +\infty$. Namely, using (3.9), one can easily prove the existence of the higher Stirling formulas:

$$\log\Gamma_n(s) = A_n(s) + B_n(s)\log s + O(1/s) \tag{3.10}$$

where A_n, B_n are polynomials of degree $\le n$. On the other hand, combining (3.6) with the Cartier-Voros asymptotic (3.3) we see that

$$\log\Gamma_n(s) = \sum_{j=0}^{n}(-1)^{j+n}\beta_{-j}^{(n)}\left(\log s - \sum_{i=0}^{j}\frac{1}{i}\right)\frac{s^j}{j!} + P_n(s) + O(1/s). \tag{3.11}$$

Therefore, we can express $P_n(s)$ and $P_n(1) = (-1)^{n-1}\log\Delta_n(1)$ via the coefficients of Stirling's formula.

For example, (3.10) for $n=2$ is:

$$\log\Gamma_2(s) = \left(\frac{s^2}{2} - s + \frac{5}{12}\right)\log s - \frac{3}{4}s^2 + \left(\frac{1}{2} - \log\sqrt{2\pi}\right)s \tag{3.12}$$
$$- \log A + \frac{1}{12} - \log\sqrt{2\pi} + O(s^{-1})$$

where A is "Kinkelin's constant:

$$\log A = \lim_{s\to\infty}\left[\log(1^1\cdot 2^2\cdots s^s) - \left(\frac{s^2}{2} + \frac{s}{2} + \frac{1}{12}\right)\log s + \frac{s^2}{4}\right]$$
$$= 1.28242713\ldots$$

for which Voros gave the following expression:

$$\log A = -\zeta'(-1) + \frac{1}{12} = -\frac{\zeta'(2)}{2\pi^2} + \frac{1}{12}(\log 2\pi + \gamma).$$

3.3.2 Gauss formula. For all $n \ge 1$ and $N \ge 1$ we have

$$\prod_{a_i=0,1,\ldots,N-1}\Delta_n\left(s + \frac{a_1+\cdots+a_n}{N}\right) = N^{-\zeta_{\Lambda_n}(s;0)}\Delta_n(Ns), \tag{3.9}$$

and similarly for Γ_n.

Proof: In fact,

$$\prod_{a_i \bmod N}\ \prod_{k_1,\ldots,k_n=0}^{\infty}\left(s + k_1 + \frac{a_1}{N} + \cdots + k_n + \frac{a_n}{N}\right)$$
$$= \prod_{a_i,k_i}\frac{1}{N}(Ns + a_1 + Nk_1 + \cdots + a_n + Nk_n)$$

It remains to apply (2.5). The polynomial $\zeta_{\Lambda_n}(s;0)$ is given by (3.2).

3.4 Gamma factor of Selberg's zeta.

M.-F. Vignéras was the first who identified the factor "at infinity" of Selberg's zeta function as a monomial in Γ_1 and Γ_2 ([36]). We will deduce this identification here from the formula of [5] expressing zeta as the product of two determinants.

Recall that the Selberg zeta of a compact complex Riemann surface X can be introduced either in terms of the choice of comples uniformization of X, or in terms of its Riemannian geometry. The latter description runs as follows. Choose a metric ds^2 with the constant curvature -1 in the conformal class of the given complex structure (this is possible precisely when the genus g of X is ≥ 2). For a primitive closed geodesic p on X put $N(p) = e^{-\ell(p)}$, where $\ell(p)$ is the positive length of p. Put

$$Z(X,s) = \prod_{p}\prod_{k=0}^{\infty}\left(1 - N(p)^{-s-k}\right). \tag{3.10}$$

3.4.1 Theorem: (Cartier, Sarnak, Voros). *$Z(X,s)$ is an entire function of order 2 and*

(3.11)

$$Z(X,s) = \left\{\exp\left((s-\frac{1}{2})^2\right) DET\left((-\Delta_{S^2}+\frac{1}{4})^{\frac{1}{2}} - \frac{1}{2} + s\right)\right\}^{2g-2} \cdot DET\Big((-\Delta_X + s^2 - s)\Big)$$

Here Δ_X is the Laplace operator on functions on X, and Δ_{S^2} is the Laplace operator on the sphere of constant curvature 1; the square root $(-\Delta_{s^2}+\frac{1}{4})^{\frac{1}{2}}$ is self-adjoint and positive.

Corollary: *Put*

$$\Gamma^*(s) = \exp\left(\left((s-\frac{1}{2})^2 - P_1(s) - 2P_2(s)\right)\Gamma_1(s)\Gamma_2(s)\right)^{2-2g}. \tag{3.11}$$

Then

$$\Gamma^*(s)Z(X,s) = DET\Big((-\Delta_X + s^2 - s)\Big) \tag{3.12}$$

is an entire function of order 2 and invariant with respect to $s \to 1-s$.

To deduce this Corollary from the Theorem, it suffices to remark that the spectrum of $-\Delta_{s^2}$ is $\{j(j+1)$ *with multiplicity* $2j+1 \mid j \geq 0\}$. Therefore, the spectrum of $(-\Delta_{s^2}+\frac{1}{4})^{\frac{1}{2}} - \frac{1}{2}$ is $\{j$ *with multiplicity* $2j+1 \mid j \geq 0\}$, and

$$\mathrm{DET}\left(\left(-\Delta_{s^2}+\frac{1}{4}\right)^{\frac{1}{2}} - \frac{1}{2} + s\right) = \left(\prod_{j\geq 0}(s+j)^{j+1}\right)^2 \Big/ \prod_{j\geq 0}(s+j)$$
$$= \Delta_2(s)^2/\Delta_1(s) = \exp(-2P_2(s))\Gamma_2(s)^2\exp(-P_1(s))\Gamma_1(s).$$

3.5 Comparison between Selberg's zeta and number theoretical zetas.

Let k be a number field. Consider the Dedekind zeta function

$$\zeta_k(s) = \prod_{\mathfrak{p}}\left(1 - N(\mathfrak{p})^{-s}\right)^{-1} \tag{3.13}$$

where the product is taken over all prime ideals $\mathfrak{p}$ of k.

3.5.1 Comparing Euler factors. Selberg himself and most authors after him compared directly (3.10) and (3.13). Cohen and Lenstra [6], however, noticed that it is interesting to study the modified Euler product in the number theoretical case as well (see 3.5 below). They defined

$$\zeta_{\mathrm{CL},k}(s) = \prod_{\mathfrak{p}}\prod_{k=0}^{\infty}\left(1 - N(\mathfrak{p})^{-s-k}\right)^{-1}, \tag{3.14}$$

the product converging for $Re(s) > 1$. From our viewpoint, of course,

$$\zeta_{\mathrm{CL},k}(s) \sim \Delta_1(s) \otimes \zeta_k(s).$$

3.5.2 Comparing gamma-factors. Since the gamma factor for $\zeta_k(s)$ is (up to $\exp(Q(s))$ $\Delta_1(s/2)^{-r_1}\Delta_1(s)^{r_2}$ where r_1 (resp. r_2) is the number of real (resp. complex) places of k, the gamma factor for $\zeta_{\mathrm{CL},k}(s)$ must be

$$\Delta_1(s) \otimes \left[\Delta_1(s_2)^{-r_1}\Delta_1(s)^{-r_2}\right] = \Delta_1\left(\frac{s-1}{2}\right)^{r_1/2}\Delta_2(s)^{-\frac{r_1+2r_2}{2}} \tag{3.16}$$

This indeed has the same structure as (3.11).

However, zeroes of (3.15) are not concentrated in a critical strip, whereas (3.14) satisfies the Riemann type conjecture. There is one more disturbing discrepancy.

If one compares (3.12) with (1.4) and (1.5), one sees that $\Gamma^*(s)$ corresponds to the product of the actual Euler factor at infinity and of the (inverse) zeta function of $\mathbf{P}^1_{\mathbf{F}_q}$ (resp. "absolute" $\mathbf{P}^1_{\mathbf{F}_1}$) ... I am not quite sure how to break $\Gamma^*(s)$ into respective factors. However, the Cartier-Voros representation (3.11) directly involves $-\Delta_{S^2}$, and $S^2 = \mathbf{P}^1_{\mathbf{C}}$. To understand which interpretation is more enlightening is an interesting challenge.

3.5.3 Cohen-Lenstra formula. Cohen and Lenstra proved that

$$\zeta_{CL,k}(s) = \sum_G \frac{1}{|\mathrm{Aut}G|} \frac{1}{|G|^{s-1}} \tag{3.17}$$

where G runs over isomorphism classes of finite A_k-modules, (A_k being the ring of integers of k), that is over torsion coherent sheaves on Spec A_k. Is there a similar interpretation of (3.13) in terms of, say, (complexes of) $\mathcal{D}$-modules on X?

The logic that led Cohen and Lenstra to introduce the r.h.s. of (3.17) is very interesting. Consider for simplicity the case $k = \mathbf{Q}$, so that G in (3.17) runs over finite abelian groups up to isomorphism. We can imagine this set as a statistical ensemble in which $1/|\mathrm{Aut}G|$ is the weight of G. This prescription does not quite define a probability measure since $\sum 1/|\mathrm{Aut}G|$ diverges but it allows us to average many interesting functions of groups:

$$< f >_{\mathbf{groups}} := \lim_{B\to\infty} \frac{\sum_{|G|\le B} f(G)/|\mathrm{Aut}\ G|}{\sum_{|G|\le B} |\mathrm{Aut}G|^{-1}}. \tag{3.18}$$

The representation (3.17) can then be effectively used to calculate (3.18), e.g. if $f(G) = 1$ for cyclic G, $f(G) = 0$ otherwise, we get the probability for a random group in our ensemble to be cyclic which is

$$\left(3\zeta(6)\prod_{i\ge 4}\zeta(i)\cdot\prod_{i\ge 1}(1-2^{-i})\right)^{-1} \simeq 0.977575.$$

Cohen and Lenstra conjectured that (the odd part of) the class groups of imaginary quadratic fields have exactly this distribution, that is, for a reasonable class of functions, the average

$$< f >_{\mathbf{fields}} := \lim_{H\to\infty} \frac{\sum_{\mathrm{disc}K\le H} f(\mathrm{Cl}^{\mathrm{odd}}(K))}{\sum_{\mathrm{disc}K\le H} 1} \tag{3.19}$$

coincides with (3.18) (where K runs over imaginary quadratic extensions of $\mathbf{Q}$). No theoretical expression for (3.19) is known, however. so that the comparison of (3.18) and (3.19) was made by using a computer.

It would be important to extend (3.17) and (3.18) to more general categories. For example, generalizations of (3.19) to real quadratic fields experimentally exhibit a very different behaviour of (3.18), and Cohen and Lenstra explain it by presence of units and change the definition of (3.18) taking into account the rank of the unit group. However, it would be more appropriate to study the statistics of the Arakelov type Picard group of K which is an extension of CL(K) by the dual torus of the unit group (this would conform to the classical Dirichlet formula which involves the product of the class number by the regulator). It remains to find an analog of the distribution (3.18) on a class of compact abelian groups endowed with the appropriate additional structure (say, an additional lattice in the character space).

A toy model is that of the category of finite dimensional vector spaces: replacing $|G|$ by $e^{z\dim G}$, and $|\mathrm{Aut}G|$ by $e^{t(\dim G)^2}$, we get a theta function $\sum_{n=0}^{\infty} e^{-tn^2-zn}$ as an analog of (3.17). There are also interesting versions of this construction for various representation categories.

3.5.3 Comparing the explicit formulas to the trace formula. This analogy, of course, dates back to Selberg as well. A major puzzle is, however, that explicit formulas are derived from the analytical properties of the number theoretical zetas, whereas in Selberg's theory the argument goes exactly in the reverse direction: one starts with the trace formula and then derives the analytic properties of the zeta function by applying the trace formula to appropriate test functions. The trace formula itself is proved in two steps (at least in the absence of continuous spectrum): a) working in the Lobachevsky plane H covering X (or more general symmetric spaces) one establishes that integral operators on H whose kernel $k(x,y)$ depend only on the distance between x, y are actually functions of the invariant Laplace operator Δ_X; b) one restricts these operators on an appropriate space of $\pi_1(X)$-invariant functions and calculates their traces in both representations. The Δ_X-representation gives a sum over the spectrum of Δ_X, whereas the integral operator representation leads to a sum over $\pi_1(X)$ of an integral transform of the same test function, which is formally transformed to a sum over primitive elements of $\pi_1(X)$.

A similar proof of explicit formulas is highly desirable, cf. interesting suggestions of D. Goldfeld [16] and Don Zagier [38].

§4. A functional equation for $\Gamma_m(s)$ and polylogarithms:

4.1 Multiple sine function.

In this section, we generalize the classical equation $\Gamma_1(s)\Gamma_1(-s) = -\frac{\pi}{s \sin \pi s}$, following N. Kurokawa [24] and M. Rovinskii [32]. The statement takes a neat form if instead of considering $\Delta_r(s) = \prod_{k=0}^{\infty}(s+k)^{h_r(k)}$, $h_r(k) = \binom{r+k-1}{r-1}$, one considers the following function:

$$(4.1)\qquad \prod\nolimits_r(s) := \prod_{k=1}^{\infty}\left[\left(1+\frac{s}{k}\right)\exp\left(-\frac{s}{k}+\frac{s^2}{2k}-\cdots\pm\frac{s^r}{k^r}\right)\right]^{k^{r-1}}, \quad r \geq 2.$$

Clearly, $\prod_r(s) = \exp(Q_r(s))\prod_{k=1}^{\infty}(s+k)^{k^{r-1}}$ for a polynomial $Q_r(s)$ so that $\Delta_r(s)$ is a monomial in $\prod_t(s)$, $t \leq r$, s, and $\exp(Q(s))$.

4.1.1 Theorem: *For $r \geq 2$, $Im(s) < 0$, we have*

(4.2)

$$\prod\nolimits_r(s)^{(-1)^{r-1}}\prod\nolimits_r(-s) = \exp\left(-\frac{(r-1)!}{(2\pi i)^{r-1}}\sum_{k=0}^{r-1}\frac{(2\pi i)^k}{k!}s^k Li_{r-k}\left(e^{-2\pi i s}\right) + \right.$$
$$\left. + \frac{\pi i}{r}s^r - \frac{s^{r-1}}{r-1} + \frac{(r-1)!}{(2\pi i)^{r-1}}\zeta(r)\right),$$

where

$$Li_r(z) = \sum_{m=1}^{\infty}\frac{z^m}{m^r} \qquad for \qquad |z| < 1.$$

Corollary: *The r.h.s. of (4.2) admits an analytic continuation as a meromorphic (even r) or entire (odd r) function of s, whereas the functions $Li_r(s)$ individually are infinitely ramified at 0, 1, ∞.*

This property was a motivation for [32]. N. Kurokawa calls (4.2) the multiple sine function.

Proof: Directly from (4.1), we find

$$\frac{d}{ds}\log\left[\prod\nolimits_r(s)^{(-1)^{r-1}}\prod\nolimits_r(-s)\right] =$$
$$= \sum_{k=1}^{\infty}k^{r-1}\left[\frac{1}{s-k}+\frac{(-1)^{r-1}}{s+k}+\frac{1}{k}\sum_{j=1}^{r}\left(\frac{j}{k}\right)^{j-1}(1+(-1)^{j+r-1})\right]$$
$$= s^{r-1}\sum_{k=1}^{\infty}\frac{2s}{s^2-k^2} = s^{r-1}\left[\pi\cot(\pi s)-\frac{1}{s}\right]$$

Now, $\prod_r(s)^{(-1)^{r-1}}\prod_r(-s)$ is holomorphic at Im $s < 0$ and equals 1 at $s = 0$. Therefore

$$\prod\nolimits_r(s)^{(-1)^{r-1}}\prod\nolimits_r(-s) = \exp\left(-\frac{s^{r-1}}{r-1}+\int_0^s u^{r-1}\pi\cot(\pi u)\,du\right).$$

Denote by $I(s)$ the integral on the r.h.s., and calculate it by putting $u = ts$, $0 \leq t \leq 1$ and using the following formulas:

$$\cot(\pi ts) = i\left(1+2\sum_{m=1}^{\infty}e^{-2\pi imst}\right) \quad \text{for} \quad \text{Im}s < 0,\ t > 0,$$

$$\int_0^1 t^{r-1}e^{\alpha t}\,dt = (-1)^{r-1}(r-1)!\frac{e^{\alpha}}{\alpha^r}\left(\sum_{k=0}^{r-1}\frac{(-1)^k}{k!}\alpha^k - e^{-\alpha}\right), \quad \alpha \in \mathbb{C}^*.$$

We get

$$I(s) = i\pi s^{r-1}\int_0^1 t^{r-1}\left(1+2\sum_{m=1}^{\infty}e^{-2\pi imst}\right)dt$$
$$= -\frac{(r-1)!}{(2\pi i)^{r-1}}\sum_{k=0}^{r-1}\frac{(2\pi i)^k}{k!}s^k\,\mathrm{Li}_{r-k}(e^{-2\pi is}) + \frac{\pi i}{r}s^r + \frac{(r-1)!}{(2\pi i)^{r-1}}\zeta(r).$$

4.2 Zagier's conjecture.

Let F be an algebraic number field, $K_m(F)$ its m–th algebraic K–group. We will generally write $A_{\mathbf{Q}}$ for $A \otimes_{\mathbf{Z}} \mathbf{Q}$. Let r_1 (resp. r_2) be the number of real (resp. complex) places of F. Put $d_n = r_1 + r_2$ (resp. r_2) for n odd (resp. n even). A. Borel [4] proved that $K_{2n-1}(F) \cong \mathbf{Z}^{d_n} \oplus \{a\ finite\ group\}$ for $n \geq 2$. Actually, this result was a by-product of properties of Borel's regulator map $reg^{\sigma} : K_{2n-1}(F) \to \mathbf{C}/(2\pi i)^n\mathbf{R}$ defined for any embedding $\sigma : F \hookrightarrow \mathbf{C}$.

P. Deligne and A. Beilinson gave a more conceptual construction of reg and generalized it to K–groups of schemes over $\mathbf{Q}$. In particular, it implies the existence of a refined regulator

(4.3) $$reg^{\sigma} : K_{2n-1}(F) \to \mathbf{C}/(2\pi i)^n\mathbf{Q} == Ext^1(\mathbf{Q}(0), \mathbf{Q}(n))$$

where the Ext–groups are calculated in the category of mixed Hodge structures ([3], [21]).

Don Zagier (see [39] and references therein) suggested a formula for the calculation of reg^{σ}. It involves a (partly conjectural) representation of $K_{2n-1}(F)_{\mathbf{Q}}$ as a subquotient of the cycle space $\mathbf{Q}[\mathbf{P}^1(F)]$. Without entering into all the details of this beautiful and complex picture (for which see [39], [3], [17], [18]) we will give the bare bones of what is relevant to us here, following [11].

Define a subspace $A_n(F) \subset \mathbf{Q}[\mathbf{P}^1 \setminus \{0, 1, \infty\}]$ by the following conditions: for $\lambda_\alpha \in \mathbf{Q}, z_\alpha \in F, \{z_\alpha\}_n$ = image of z_α in $\mathbf{Q}[\mathbf{P}^1(F)]$, $\sum \lambda_\alpha \{z_\alpha\}_n \in A_n(F)$ iff:

a). For every homomorphism $v : F^* \to \mathbf{Q}$, we have

$$\sum_\alpha \lambda_\alpha v(z_\alpha)^{n-2}(1 - z_\alpha) \wedge z_\alpha = 0 \text{ in } (\wedge^2 F)_{\mathbf{Q}}.$$

b). For every $2 \leq m < n$ and every complex embedding $\sigma : F \hookrightarrow \mathbf{C}$, we have

$$\sum_\alpha \lambda_\alpha v(z_\alpha)^{n-m} D_m(\sigma z_\alpha) = 0$$

where

$$D_m(z) = i \sum_l \frac{b_l}{l!} \log|z|^{2l} \Re_m(Li_{m-l}(z)),$$

$\Re_m = \Re$ for m odd, $\Im$ for m pair; Li_m and b_l are defined by (4.5) below.

4.2.1 Conjecture: *There exists a surjective map $\rho_n : A_n(F) \to K_{2n-1}(F)$ such that*

(4.4) $$reg^{\sigma} \rho_n(\sum \lambda_\alpha \{z_\alpha\}_n) \equiv \sum \lambda_\alpha \Lambda_n(z_\alpha) \mod (2\pi i)^n\mathbf{Q}.$$

where

(4.5) $$\Lambda_n(z) := \sum_{j=0}^{n-1} b_j \frac{(\log z)^j}{j!} Li_{n-j}(z),$$

$$Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k} \quad for \quad |z| < 1,\ k \geq 1,$$

$$\sum_{m=0}^{\infty} b_m \frac{t^m}{m!} = \frac{t}{e^t - 1}.$$

(Functions $\Lambda_n(z)$ are multivalued, and the correct definition of the r.h.s. of (4.4) involves a careful choice of branches. We omit the details).

Using Theorem 4.1.1, one can rewrite the r.h.s. of (4.4) in terms of multiple gammas.

4.3 Proposition: *We have*

$$\Lambda_n(z) = \sum_{a=0}^{n-1} (-1)^a \frac{b_a}{a!} (\log z)^a (2\pi i)^{r-a-1} \log N_{r-a}(z),$$

where $N_{r-a}(s)$ is a monomial in multiple gamma functions and exponents of polynomials whose values are taken at $\pm\frac{\log z}{2\pi i}$.

Proof: Put $z = e^{-2\pi i s}$, $s = -\frac{\log z}{2\pi i}$ in (4.2). We then have:

$$H_r(z) := \sum_{k=0}^{r-1} (-1)^k \frac{(\log z)^k}{k!} \mathrm{Li}_{r-k}(z) = (2\pi i)^{r-1} \log N_r(z).$$

Consider now a linear combination with undetermined coefficients x_a :

$$\sum_{a=0}^{r-1} x_a (\log z)^a H_{r-a}(z) = \Lambda_r(z) := \sum_{j=0}^{r-1} b_j \frac{(\log z)^j}{j!} \mathrm{Li}_{r-j}(z).$$

Since the l.h.s. is

$$\sum_{a=0}^{r-1} x_a (\log z)^a \sum_{b=0}^{r-1} (-1)^b \frac{(\log z)^b}{b!} \mathrm{Li}_{r-a-b}(z)$$

$$= \sum_{j=0}^{r-1} \left(\sum_{\substack{a+b=j \\ a,b \geq 0}} x_a \frac{(-1)^b}{b!} \right) (\log z)^j \mathrm{Li}_{r-j}(z),$$

we want

$$\left(\sum_{a=0}^{\infty} x_a t^a\right)\left(\sum_{b=0}^{\infty}\frac{(-1)^b}{b!}t^b\right) = \sum_{j=0}^{\infty}\frac{b_j}{j!}t^j,$$

and one easily sees that $x_a = (-1)^a \frac{b_a}{a!}$.

§5. Concluding remarks:

5.1 Selberg's class.

In the absence of the geometric framework for the hypothetical absolute motives, one can at least hope to understand the respective class of zeta functions.

For motives over $\mathbf{F}_q$, it has a nice description in terms of Weil's numbers. *A Weil number of q-weight $w \geq 0$* is an algebraic integer whose conjugates α all verify the condition $|\alpha| = q^{w/2}$. Zeta functions of motives of pure weight w over $\mathbf{F}_q$ correspond to polynomials $\prod_{\alpha\in M}(1-\alpha T)$, $T = q^{-s}$, where M runs over finite Gal $\overline{\mathbb{Q}}/\mathbb{Q}$-invariant sets of Weil's numbers. This class is stable with respect to Kurokawa's G_m-tensor product. Every zeta function admits a unique decomposition into a product of *primitive* zeta functions, corresponding to irreducible polynomials.

It would be quite important to axiomatize a similar class of entire functions stable with respect to G_a-tensor products (with an appropriate regularization) and containing arithmetical motivic zetas. As examples suggest, it must contain entire functions of an arbitrary integral growth order, which corresponds to the "Spec $\mathbb{Z}$-weight" of an absolute motive. If functions of this class admit a unique decomposition into primitive ones, the latter should correspond to irreducible absolute motives.

In [33], A. Selberg suggested one consider the following class of Dirichlet series (see Ram Murty [30] for more details): $F(s) \in \mathcal{S}$ iff it satisfies five conditions:

(i) $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ for $\mathrm{Re}(s) > 1$, $a_1 = 1$, $a_n = a_n(F)$.

(ii) $F(s)(s-1)^m$ is entire of finite order for a certain integer $m \geq 0$.

(iii) For some $Q > 0$, $\alpha_i > 0$, $\mathrm{Re}(\alpha_i) > 0$, $|w| = 1$, the function $\Phi(s) = Q^s \prod_{i=1}^{d}\Gamma(\alpha_i s + r_i)F(s)$ satisfies $\Phi(s) = w\overline{\Phi(1-\bar{s})}, |w| = 1$

(iv) $F(s) = \prod_p F_p(s)$; $\log F_p(s) = \sum_{n=1}^{\infty} b_{p^k} p^{-ks}$, $b_{p^k} = O(p^{k\theta})$ for some $\theta < \frac{1}{2}$; p runs over primes.

(v) $a_n(F) = O(n^{\epsilon})$ for every $\epsilon > 0$.

Clearly, $\mathcal{S}$ is a multiplicative monoid. A function $F \in \mathcal{S}$ is called *primitive* if it is indecomposable in $\mathcal{S}$. One can prove that every element of $\mathcal{S}$ is a product of indecomposable ones, and this representation is unique, if the following beautiful conjectures of Selberg are true (see Murty [30]):

Conjecture A. *For every $F \in \mathcal{S}$ there exists such an integer $n_F > 0$ such that*

$$\sum_{p\leq x} |a_p(F)|^2/p = n_F \log\log x + O(1).$$

Conjecture B. *The following hold:*
i) If F is primitive then $n_F = 1$.
ii) If $F \neq G$ are primitive then

$$\sum_{p\leq x}\frac{a_p(F)\overline{a_p(G)}}{p} = O(1).$$

A scalar product that lurks behind these formulas must be a shadow of an absolute motivic correspondence ring and Hodge star operators (cf. Deninger [14], 7.11).

5.2 Kurokawa's splitting.

For the tensor square of the Riemann zeta function (or rather its weight 1 part $\Gamma_{\mathbb{R}}(s)s(s-1)\zeta(s) := \xi(s)$) the prescription (1.19) is inapplicable because, e.g., 1 has infinite multiplicity. Kurokawa suggests that one split $\xi(s)$ into the product of $\xi_{\pm}(s) := \prod_{\mathrm{Im}\rho\gtrless 0}\left(1-\frac{s}{\rho}\right)e^{s/\rho}$, where ρ runs over the critical zeroes of $\zeta(s)$, and then redefine $\xi(s)^{\otimes r}$ as

$$\xi(s)^{\otimes r} := \xi_+(s)^{\otimes r}\left[\xi_-(s)^{\otimes r}\right]^{(-1)^{r-1}}, \quad (5.1)$$

by analogy with (4.2).

There remains much to be done to see whether this is a good definition. In particular, one must understand the relation of $\xi(s)^{\otimes r}$ to primes.

A work of H. Cramer and A.P. Guinand shows at least that there exist versions of explicit formulae in which summation is taken over half of the zeroes of the zeta function. We will quote here two relevant theorems, their analogues for the Selberg zeta are proved in [5].

5.2.1 Theorem: *Put*

$$V(w)=\sum_{\mathrm{Im}\rho>0} e^{\rho w}.$$

This series converges absolutely for $Im\, w>0$.

a) $V(w)$ admits a meromorphic continuation to the whole plane $\mathbb{C}$ cut from 0 to $-i\infty$, and has there first order poles at $w=\pm\log p^m$, with principal parts $\frac{\log p}{2\pi i}\frac{1}{w-\log p^m}$ and $\frac{\log p}{2\pi i p^m}\frac{1}{w+\log p^m}$ respectively.

b) Near $w=0$, we have

$$V(w)=\frac{1}{2\pi i}\left(\frac{\log w}{1-e^{-iw}}+\frac{\gamma+\log 2\pi-\frac{\pi i}{2}}{w}\right)+\frac{3}{4}+\eta i+wW(w) \tag{5.3}$$

where $\eta>0$, and $W(w)$ is single valued and regular for $|w|<\log 2$.

c) Put $U(w)=e^{-\frac{1}{2}iw}V(iw)+\frac{1}{4\pi}\frac{\log w}{\sin\frac{w}{2}}$. This function admits a single valued analytic continuation for which

$$U(w)+U(-w)=2\cos\frac{w}{2}-\frac{1}{4\cos\frac{w}{2}}.$$

Parts a) and b) are proved in [7], part c) in [20]. From (5.2) one can easily deduce that $\prod_{\mathrm{Im}\rho>0}(s-\rho)$ does not exist. More precisely:

5.2.2 Proposition: *Put $\Lambda=\{i\rho \mid Im\,\rho>0\}$. Then $\Lambda^{\otimes r}$, $r\geq 1$, is a left directed family, and*

$$\zeta_{\Lambda^{\otimes r}}(z,s):=\sum_{\mathrm{Im}\,\rho>0}\frac{1}{(s-i\rho_1-\ldots-i\rho_r)^z}$$

has a meromorphic continuation to a cut z-plane, with poles of order r at $z=0,-1,\ldots,-r$, coefficients of principal parts of which are polynomials in s.

In fact, from (5.3) one derives that as $t\to+0$,

$$V(it)=-\frac{1}{t}\left(\frac{\log t}{2\pi i}+\frac{\pi}{4}+\frac{\gamma+\log 2\pi}{2\pi}+\frac{i}{4}\right)+\frac{1}{4\pi i}\log t+O(1),$$

and it remains to apply the Mellin transform to $V(it)^r$, as in §2.

The assertion a) of Theorem 4.2 can be considered as a kind of explicit formula, but not quite since $V(z)$ does not reduce to the sum of its principle parts. However, under the assumption of Riemann's conjecture, A.P. Guinand [19] proved the following explicit formula:

5.2.3 Theorem: *Assume the function f defined on $[0,\infty)$ is an integral, $f(x)\to\infty$ as $x\to+\infty$, and $xf'(x)$ belongs to $L^p(0,\infty)$ for some p, $1<p\leq 2$. Put*

$$g(x)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(t)\cos xt\,dt.$$

If Riemann's conjecture is true then

$$\lim_{T\to\infty}\left\{\sum_{0<m\log p<T}\frac{\log p}{p^{m/2}}f(m\log p)-\int_0^T f(x)e^{x/2}\,dx\right.$$
$$\left.-f(T)\left(\sum_{0<m\log p<T}\frac{\log p}{p^{m/2}}-2e^{T/2}\right)\right\}-\frac{1}{2}\int_0^\infty f(x)\left(\frac{1}{x}-\frac{e^{-\frac{3x}{2}}}{\sinh x}\right)dx$$
$$=-\sqrt{2\pi}\lim_{T\to\infty}\left\{\sum_{0<\gamma<T}g(\gamma)-\frac{1}{2\pi}\int_0^T g(x)\log\frac{x}{2\pi}\,dx\right\},$$

where γ runs over imaginary parts of zeroes.

Similar results must be true for all functions in Selberg's class; c.f. P.X Gallagher's work "Applications of Guinand's formulas," (In: Analytic Number Theory and Diophantine Problems, Ac. Press, 1987, 135-157).

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