

Noncommutative geometry and motives (a quoi servent les endomotifs?)

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Abstract. This paper gives a short and historical survey on the theory of pure motives in algebraic geometry and reviews some of the recent developments of this theory in noncommutative geometry. The second part of the paper outlines the new theory of endomotives and some of its relevant applications in number-theory.

1 Introduction

This paper is based on three lectures I gave at the Conference on “Renormalization and Galois theories” that was held in Luminy, at the Centre International de Rencontres Mathématiques (CIRM), on March 2006. The purpose of these talks was to give an elementary overview on classical motives (pure motives) and to survey on some of the recent developments of this theory in noncommutative geometry, especially following the introduction of the notion of an *endomotive*.

It is likely to expect that the reader acquainted with the literature on motives theory will not fail to notice the allusion, in this title, to the paper [17] in which P. Deligne states that in spite of the lack of essential progresses on the problem of constructing “relevant” algebraic cycles, the techniques supplied by the theory of motives remain a powerful tool in algebraic geometry and arithmetic.

The assertion on the lack of relevant progresses on algebraic cycles seems, unfortunately, still to apply at the present time, fifteen years after Deligne wrote his paper. Despite the general failure of testing the Standard Conjectures, it is also true that in these recent years the knowledge on motives has been substantially improved by several new results and also by some unexpected developments.

Motives were introduced by A. Grothendieck with the aim to supply an intrinsic explanation for the analogies occurring among various cohomological theories in algebraic geometry. They are expected to play the role of a universal cohomological

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theory by also furnishing a linearization of the theory of algebraic varieties and in the original understanding they were expected to provide the correct framework for a successful approach to the Weil's Conjectures on the zeta-function of a variety over a finite field.

Even though the Weil's Conjectures have been proved by Deligne without appealing to the theory of motives, an enlarged and in part still conjectural theory of mixed motives has in the meanwhile proved its usefulness in explaining conceptually, some intriguing phenomena arising in several areas of pure mathematics, such as Hodge theory, K -theory, algebraic cycles, polylogarithms, L -functions, Galois representations etc.

Very recently, some new developments of the theory of motives to number-theory and quantum field theory have been found or are about to be developed, with the support of techniques supplied by noncommutative geometry and the theory of operator algebras.

In number-theory, a conceptual understanding of the main result of [10] on the interpretation proposed by A. Connes of the Weil explicit formulae as a Lefschetz trace formula over the noncommutative space of adèle classes, requires the introduction of a generalized category of motives inclusive of spaces which are highly singular from a classical viewpoint.

The problem of finding a suitable enlargement of the category of (smooth projective) algebraic varieties is combined with the even more compelling one of the definition of a generalized notion of correspondences. Several questions arise already when one considers special types of zero-dimensional noncommutative spaces, such as the space underlying the quantum statistical dynamical system defined by J. B. Bost and Connes in [6] (the BC-system). This space is a simplified version of the adèles class space of [10] and it encodes in its group of symmetries, the arithmetic of the maximal abelian extension of \mathbb{Q} .

In this paper I give an overview on the theory of endomotives (algebraic and analytic). This theory has been originally developed in the joint paper [7] with A. Connes and M. Marcolli and has been applied already in our subsequent work [11]. The category of endomotives is the minimal one that makes it possible to understand conceptually the role played by the absolute Galois group in several dynamical systems that have been recently introduced in noncommutative geometry as generalizations of the BC-system, which was our motivating and prototype example.

The category of endomotives is a natural enlargement of the category of Artin motives: the objects are noncommutative spaces defined by semigroup actions on projective limits of Artin motives. The morphisms generalize the notion of algebraic correspondences and are defined by means of étale groupoids to account for the presence of the semigroup actions.

Endomotives carry a natural Galois action which is inherited from the Artin motives and they have both an algebraic and an analytic description. The latter is particularly useful as it provides the data of a quantum statistical dynamical system, via the implementation of a *canonical time evolution* (a one-parameter family of automorphisms) which is associated by the theory of M. Tomita (*cf.* [36]) to an initial state (probability measure) assigned on an analytic endomotive. This

is the crucial new development supplied by the theory of operator-algebras to a noncommutative C^* -algebra and in particular to the algebra of the BC-system.

The implication in number-theory is striking: the time evolution implements on the dual system a scaling action which combines with the action of the Galois group to determine on the cyclic homology of a suitable noncommutative motive associated to the original endomotive, a characteristic zero analog of the action of the Weil group on the étale cohomology of an algebraic variety. When these techniques are applied to the endomotive of the BC-system or to the endomotive of the adèles class space, the main implication is the spectral realization of the zeroes of the corresponding L -functions.

These results supply a first answer to the question I raised in the title of this paper (a *quoi servent les endomotifs?*). An open and interesting problem is connected to the definition of a higher dimensional theory of noncommutative motives and in particular the introduction of a theory of noncommutative elliptic motives and modular forms. A related problem is of course connected to the definition of a higher dimensional theory of geometric correspondences. The comparison between algebraic correspondences for motives and geometric correspondences for noncommutative spaces is particularly easy in the zero-dimensional case, because the equivalence relations play no role. In noncommutative geometry, algebraic cycles are naturally replaced by bi-modules, or by classes in equivariant KK -theory. Naturally, the original problem of finding “interesting” cycles pops-up again in this topological framework: a satisfactory solution to this question seems to be one of the main steps to undertake for a further development of these ideas.

2 Classical motives: an overview

The theory of motives in algebraic geometry was established by A. Grothendieck in the 1960s: 1963-69 (*cf.* [35],[21]). The foundations are documented in the unpublished manuscript [19] and were discussed in a seminar at the Institut des Hautes Études Scientifiques, in 1967. This theory was conceived as a fundamental machine to develop Grothendieck’s “long-run program” focused on the theme of the connections between geometry and arithmetic.

At the heart of the philosophy of motives sit Grothendieck’s speculations on the existence of a universal cohomological theory for algebraic varieties defined over a base field k and taking values into an abelian, tensor category. The study of this problem originated as the consequence of a general dissatisfaction connected to an extensive use of topological methods in algebraic geometry, with the result of producing several but insufficiently related cohomological theories (Betti, de-Rham, étale etc.). The typical example is furnished by a family of homomorphisms $H_{\text{et}}^i(X, \mathbb{Q}_\ell) \rightarrow H_{\text{et}}^i(Y, \mathbb{Q}_\ell)$ connecting the groups of étale cohomology of two (smooth, projective) varieties, as the prime number ℓ varies, which are not connected, in general, by any sort of (canonical) relation.

The definition of a contravariant functor (functor of motivic cohomology)

$$h : \mathcal{V}_k \rightarrow \mathcal{M}_k(\mathcal{V}_k), \quad X \mapsto h(X)$$

from the category \mathcal{V}_k of projective, smooth, irreducible algebraic varieties over k to a semi-simple abelian category of pure motives $\mathcal{M}_k(\mathcal{V}_k)$ is also tied up with the definition of a universal cohomological theory through which every other classical, cohomology H^* (here understood as contravariant functor) should factor by means of the introduction of a fiber functor (realization \otimes -functor) ω connecting $\mathcal{M}_k(\mathcal{V}_k)$ to the abelian category of (graded) vector-spaces over \mathbb{Q}

$$\begin{array}{ccc} \mathcal{V}_k & \xrightarrow{H^*} & grVect_{\mathbb{Q}} \\ & \searrow h & \uparrow \omega \\ & & \mathcal{M}_k(\mathcal{V}_k) \end{array}$$

Following Grothendieck's original viewpoint, the functor h should implement the sought for mechanism of compatibilities (in étale cohomology) and at the same time it should also describe a universal linearization of the theory of algebraic varieties.

The definition of the category $\mathcal{M}_k(\mathcal{V}_k)$ arised from a classical construction in algebraic geometry which is based on the idea of extending the collection of algebraic morphisms in \mathcal{V}_k by including the (algebraic) *correspondences*. A correspondence between two objects X and Y in \mathcal{V}_k is a multi-valued map which connects them. An algebraic correspondence is defined by means of an algebraic cycle in the cartesian product $X \times Y$. The concept of (algebraic) correspondence in geometry is much older than that of a motive: it is in fact already present in several works of the Italian school in algebraic geometry (*cf.* Severi's theory of correspondences on algebraic curves).

Grothendieck's new intuition was that the whole philosophy of motives is regulated by the theory of (algebraic) correspondences:

"...J'appelle motif sur k quelque chose comme un groupe de cohomologie ℓ -adique d'un schema algébrique sur k , mais considérée comme indépendant de ℓ , et avec sa structure entière, ou disons pour l'instant sur \mathbb{Q} , déduite de la théorie des cycles algébriques..." (*cf.* [15], Lettre 16.8.1964).

Motives were envisioned with the hope to explain the intrinsic relations between integrals of algebraic functions in one or more complex variables. Their ultimate goal was to supply a machine that would guarantee a generalization of the main results of Galois theory to systems of polynomials equations in several variables. Here, we refer in particular to a higher-dimensional analog of the well-known result which describes the linearization of the Galois-Grothendieck correspondence for the category \mathcal{V}_k^o of étale, finite k -schemes

$$\mathcal{V}_k^o \xrightarrow{\sim} \{\text{finite sets with } \text{Gal}(\bar{k}/k)\text{-action}\}, \quad X \mapsto X(\bar{k})$$

by means of the equivalence between the category of Artin motives and that of the representations of the absolute Galois group.

In the following section we shall describe how the fundamental notions of the theory of motives arose from the study of several classical problems in geometry and arithmetic.

2.1 A first approach to motives

A classical problem in algebraic geometry is that of computing the solutions of a finite set of polynomial equations

$$f_1(X_1, \dots, X_m) = 0, \dots, f_r(X_1, \dots, X_m) = 0$$

with coefficients in a finite field \mathbf{F}_q . This study is naturally formalized by introducing the generating series

$$\zeta(X, t) = \exp\left(\sum_{m \geq 1} \nu_m \frac{t^m}{m}\right) \tag{2.1}$$

which is associated to the algebraic variety $X = V(f_1, \dots, f_m)$ that is defined as the set of the common zeroes of the polynomials f_1, \dots, f_r .

Under the assumption that X is smooth and projective, the series (2.1) encodes the complete information on the number of the rational points of the algebraic variety, through the coefficients $\nu_m = |X(\mathbf{F}_{q^m})|$. The integers ν_m supply the cardinality of the set of the rational points of X , computed in successive finite field extensions \mathbf{F}_{q^m} of the base field \mathbf{F}_q .

Intersection theory furnishes a general way to determine the number ν_m as intersection number of two algebraic cycles on the cartesian product $X \times X$: namely the diagonal Δ_X and the graph Γ_{Fr^m} of the m -th iterated composite of the Frobenius morphism on the scheme (X, \mathcal{O}_X) :

$$Fr : X \rightarrow X; \quad Fr(P) = P, \quad f(\underline{x}) \mapsto f(\underline{x}^q), \quad \forall f \in \mathcal{O}_X(U), \forall U \subset X \text{ open set.}$$

The Frobenius endomorphism is in fact an interesting example of *correspondence*, perhaps the most interesting one, for algebraic varieties defined over finite fields. As a correspondence it induces a commutative diagram

$$\begin{array}{ccc} H^*(X \times X) & \xrightarrow{-\cap \Gamma_{Fr}} & H^*(X \times X) \\ \uparrow p_1^* & & \downarrow (p_2)_* \\ H^*(X) & \xrightarrow{Fr^*} & H^*(X) \end{array} \tag{2.2}$$

in étale cohomology. Through the commutativity of the above diagram one gets a way to express the action of the induced homomorphism in cohomology, by means of the formula

$$Fr^*(c) = (p_2)_*(p_1^*(c) \cap \Gamma_{Fr}), \quad \forall c \in H^*(X),$$

where $p_i : X \times X \rightarrow X$ denote the two projection maps. Here, ‘algebraic’ refers to the algebraic cycle $\Gamma_{Fr} \subset X \times X$ that performs such a correspondence.

For particularly simple algebraic varieties, such as projective spaces \mathbf{P}^n , the computation of the integers ν_m can be done by applying an elementary combinatorial argument based on the set-theoretical description of the space $\mathbf{P}^n(k) = k^{n+1} \setminus \{0\}/k^\times$ ($k = \text{any field}$). This has the effect to produce the interesting description

$$|\mathbf{P}^n(\mathbf{F}_{q^m})| = \frac{q^{m(n+1)} - 1}{q^m - 1} = 1 + q^m + q^{2m} + \cdots + q^{mn}. \quad (2.3)$$

This decomposition of the set of the rational points of a projective space was certainly a first source of inspiration in the process of formalizing the foundations of the theory of motives. In fact, one is naturally led to wonder on the causality of the decomposition (2.3), possibly ascribing such a result to the presence of a cellular decomposition on the projective space which induces a related break-up on the set of the rational points. Remarkably, A. Weil proved that a similar formula holds also in the more general case of a smooth, projective algebraic curve C/\mathbf{F}_q of genus $g \geq 0$. In this case one shows that

$$|C(\mathbf{F}_{q^m})| = 1 - \sum_{i=1}^{2g} \omega_i^m + q^m; \quad \omega_i \in \bar{\mathbb{Q}}, \quad |\omega_i| = q^{1/2}. \quad (2.4)$$

These results suggest that (2.3) and (2.4) are the manifestation of a deep and intrinsic structure that governs the geometry of algebraic varieties.

The development of the theory of motives has in fact shown to us that this structure reveals itself in several contexts: topologically, manifests its presence in the decomposition of the cohomology $H^*(X) = \bigoplus_{i \geq 0} H^i(X)$, whereas arithmetically it turns out that it is the same structure that controls the decomposition of the series (2.1) as a rational function of t :

$$\zeta(X, t) = \frac{\prod_{i \geq 0} \det(1 - tFr^*|H_{et}^{2i+1}(X))}{\prod_{i \geq 0} \det(1 - tFr^*|H_{et}^{2i}(X))}.$$

This is in fact a consequence of the description of the integers ν_m supplied by the Lefschetz-Grothendieck trace formula (*cf.* [22])

$$|X(\mathbf{F}_{q^m})| = \sum_{i \geq 0} (-1)^i \text{tr}((Fr^m)^*|H_{et}^i(X)).$$

2.2 Grothendieck’s pure motives

Originally, Grothendieck proposed a general framework for a so called category of *numerically effective motives* $\mathbf{M}(k)_{\mathbb{Q}}$ over a field k and with rational coefficients. This category is defined by enlarging the category \mathcal{V}_k of smooth, projective algebraic varieties over k (and algebraic morphisms) by following the so-called procedure of pseudo-abelian envelope. This construction is performed in two steps: at first one enlarges the set of morphisms of \mathcal{V}_k by including (rational) algebraic

correspondences of degree zero, modulo numerical equivalence, then one performs a pseudo-abelian envelope by formally including among the objects, kernels of idempotent morphisms.

Let us assume for simplicity that the algebraic varieties are irreducible (the general case is then deduced from this, by additivity). For any given $X, Y \in \text{Obj}(\mathcal{V}_k)$, one works with correspondences $f : X \dashrightarrow Y$ which are elements of codimension equal to $\dim X$ in the rational graded algebra

$$A^*(X \times Y) = C^*(X \times Y) \otimes \mathbb{Q} / \sim_{num}$$

of algebraic cycles modulo *numerical equivalence*. We recall that two algebraic cycles on an algebraic variety X are said to be numerically equivalent $Z \sim_{num} W$, if

$$\deg(Z \cdot T) = \deg(W \cdot T), \tag{2.5}$$

for any algebraic cycle T on X . Here, by $\deg(V)$ we mean the degree of the algebraic cycle $V = \sum_{\text{finite}} m_\alpha V_\alpha \in C^*(X)$.

The degree defines a homomorphism from the free abelian group of algebraic cycles $C^*(X) = \bigoplus_i C^i(X)$ to the integers. On the components $C^i(X)$, the map is defined as follows

$$\deg : C^i(X) \rightarrow \mathbb{Z}, \quad \deg(V) = \begin{cases} \sum_\alpha m_\alpha & \text{if } i = \dim X, \\ 0 & \text{if } i < \dim X. \end{cases}$$

The symbol ‘ \cdot ’ in (2.5) refers to the intersection product structure on $C^*(X)$, which is well-defined under the assumption of proper intersection. If $Z \cap T$ is proper (*i.e.* $\text{codim}(Z \cap T) = \text{codim}(Z) + \text{codim}(T)$), then intersection theory supplies the definition of an intersection cycle $Z \cdot T \in C^*(X)$. Moreover, the intersection product is commutative and associative whenever is defined.

Passing from the free abelian group $C^*(X)$ to the quotient $C^*(X) / \sim$, modulo a suitable equivalence relation on cycles, allows one to use classical results of algebraic geometry (so called Moving Lemmas) which lead to the definition of a ring structure. One then defines intersection cycle classes in general, even when cycles do not intersect properly, by intersecting equivalent cycles which fulfill the required geometric property of proper intersection.

It is natural to guess that the use of the numerical equivalence in the original definition of the category of motives was motivated by the study of classical constructions in enumerative geometry, such as for example the computation of the number of the rational points of an algebraic variety defined over a finite field. One of the main original goals was to show that for a suitable definition of an equivalence relation on algebraic cycles, the corresponding category of motives is *semi-simple*. This means that the objects M in the category decompose, following the rules of a *theory of weights* (*cf.* section 2.3), into direct *factors* $M = \bigoplus_i M_i(X)$, with $M_i(X)$ simple (*i.e.* indecomposable) motives associated to smooth, projective algebraic varieties. The importance of achieving such a result is quite evident if one seeks, for example, to understand categorically the decomposition $H^*(X) = \bigoplus_i H^i(X)$ in cohomology, or if one wants to recognize the role of motives in the factorization of zeta-functions of algebraic varieties.

Grothendieck concentrated his efforts on the numerical equivalence relation which is the coarsest among the equivalence relations on algebraic cycles. So doing, he attacked the problem of the semi-simplicity of the category of motives from the easiest side. However, despite a promising departing point, the statement on the semi-simplicity escaped all his efforts. In fact, the result he was able to reach at that time was dependent on the assumption of the *Standard Conjectures*, two strong topological statements on algebraic cycles. The proof of the semi-simplicity of the category of motives for numerical equivalence (the only equivalence relation producing this result) was achieved only much later on in the development of the theory (*cf.* [24]). The proof found by U. Jannsen uses a fairly elementary but ingenious idea which mysteriously eluded Grothendieck's intuition as well as all the mental grasps of several mathematicians after him.

By looking at the construction of (pure) motives in perspective, one immediately recognizes the predominant role played by the morphisms over the objects, in the category $\mathbf{M}(k)_{\mathbb{Q}}$. This was certainly a great intuition of Grothendieck. This idea led to a systematic study of the properties of algebraic cycles and their decomposition by means of *algebraic projectors*, that is algebraic cycles classes $p \in A^{\dim X}(X \times X)$ satisfying the property

$$p^2 = p \circ p = p.$$

Notice that in order to make sense of the notion of a projector and more in general, in order to define a law of composition 'o' on algebraic correspondences, one needs to use the ring structure on the graded algebra A^* . The operation 'o' is defined as follows. Let us assume for simplicity, that the algebraic varieties are connected (the general case can be easily deduced from this). Then, two algebraic correspondences $f_1 \in A^{\dim X_1+i}(X_1 \times X_2)$ (of degree i) and $f_2 \in A^{\dim X_2+j}(X_2 \times X_3)$ (of degree j) compose accordingly to the following rule (bi-linear, associative)

$$A^{\dim X_1+i}(X_1 \times X_2) \times A^{\dim X_2+j}(X_2 \times X_3) \rightarrow A^{\dim X_1+i+j}(X_1 \times X_3)$$

$$(f_1, f_2) \mapsto f_2 \circ f_1 = (p_{13})_*((p_{12}^*(f_1) \cdot (p_{23})^*(f_2))).$$

In the particular case of projectors $p : X \dashrightarrow X$, one is restricted, in order to make a sense of the condition $p \circ p = p$, to use only particular types of algebraic correspondences: namely those of degree zero. These are the elements of the abelian group $A^{\dim X}(X \times X)$.

The objects of the category $\mathbf{M}(k)_{\mathbb{Q}}$ are then pairs (X, p) , with $X \in \text{Obj}(\mathcal{V}_k)$ and p a projector. This way, one attains the notion of a \mathbb{Q} -linear, pseudo-abelian, monoidal category (the \otimes -monoidal structure is deduced from the cartesian product of algebraic varieties), together with the definition of a contravariant functor

$$h : \mathcal{V}_k \rightarrow \mathbf{M}(k)_{\mathbb{Q}}, \quad X \mapsto h(X) = (X, \text{id}).$$

Here (X, id) denotes the motive associated to X and id means the trivial (*i.e.* identity) projector associated to the diagonal Δ_X . More in general, (X, p) refers to the motive $ph(X)$ that is cut-off on $h(X)$ by the (range of the) projector $p : X \dashrightarrow X$. Notice that images of projectors are formally included among the objects of $\mathbf{M}(k)_{\mathbb{Q}}$, by the procedure of the pseudo-abelian envelope. The cut-off

performed by a projector p on the space determines a corresponding operation in cohomology (for any classical Weil theory), by singling out the sub-vector space $pH^*(X) \subset H^*(X)$.

The category $\mathbf{M}(k)_{\mathbb{Q}}$ has two important basic objects: $\mathbf{1}$ and \mathbf{L} . $\mathbf{1}$ is the *unit motive*

$$\mathbf{1} = (\mathrm{Spec}(k), \mathrm{id}) = h(\mathrm{Spec}(k)).$$

This is defined by the zero-dimensional algebraic variety associated to a point, whereas

$$\mathbf{L} = (\mathbf{P}^1, \pi_2), \quad \pi_2 = \mathbf{P}^1 \times \{P\}, \quad P \in \mathbf{P}^1(k)$$

is the so-called *Lefschetz motive*. This motive determines, jointly with $\mathbf{1}$, a decomposition of the motive associated to the projective line \mathbf{P}^1

$$h(\mathbf{P}^1) = \mathbf{1} \oplus \mathbf{L}. \tag{2.6}$$

One can show that the algebraic cycles $\mathbf{P}^1 \times \{P\}$ and $\{P\} \times \mathbf{P}^1$ on $\mathbf{P}^1 \times \mathbf{P}^1$ do not depend on the choice of the rational point $P \in \mathbf{P}^1(k)$ and that their sum is a cycle equivalent to the diagonal. This fact implies that the decomposition (2.6) is canonical. More in general, it follows from the Künneth decomposition of the diagonal Δ in $\mathbf{P}^n \times \mathbf{P}^n$ by algebraic cycles $\Delta = \pi_0 + \dots + \pi_n$, (*cf.* [31] and [18] for the details) that the motive of a projective space \mathbf{P}^n decomposes into pieces (simple motives)

$$h(\mathbf{P}^n) = h^0(\mathbf{P}^n) \oplus h^2(\mathbf{P}^n) \dots \oplus h^{2n}(\mathbf{P}^n) \tag{2.7}$$

where $h^{2i}(\mathbf{P}^n) = (\mathbf{P}^n, \pi_{2i}) = (h^2(\mathbf{P}^n))^{\otimes i}$, $\forall i > 0$. It is precisely this decomposition which implies the decomposition (2.3) on the rational points, when $k = \mathbf{F}_q$!

For (irreducible) curves, and in the presence of a rational point $x \in C(k)$, one obtains a similar decomposition (non canonical)

$$h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C)$$

with $h^0(C) = (C, \pi_0 = \{x\} \times C)$, $h^2(C) = (C, \pi_2 = C \times \{x\})$ and $h^1(C) = (C, 1 - \pi_0 - \pi_2)$. This decomposition is responsible for the formula (2.4).

In fact, one can prove that these decompositions partially generalize to any object $X \in \mathrm{Obj}(\mathcal{V}_k)$. In the presence of a rational point, or more in general by choosing a positive zero-cycle $Z = \sum_{\alpha} m_{\alpha} Z_{\alpha} \in C^{\dim X}(X \times X)$ (here X is assumed irreducible for simplicity and $\dim X = d$), one constructs two rational algebraic cycles

$$\pi_0 = \frac{1}{m}(Z \times X), \quad \pi_{2d} = \frac{1}{m}(X \times Z); \quad m = \deg(Z) = \sum_{\alpha} m_{\alpha} > 0$$

which determine two projectors π_0, π_{2d} in the Chow group $CH^d(X \times X) \otimes \mathbb{Q}$ of rational algebraic cycles modulo rational equivalence. The corresponding classes in $A^d(X \times X)$ (if not zero) determine two motives $(X, \pi_0) \simeq h^0(X)$ and $(X, \pi_{2d}) \simeq h^{2d}(X)$ (*cf. e.g.* [34]).

For the applications, it is convenient to enlarge the category of effective motives by formally adding the tensor product inverse \mathbf{L}^{-1} of the Lefschetz motive: one usually refers to it as to the *Tate motive*. It corresponds, from the more refined

point of view of Galois theory, to the cyclotomic characters. This enlargement of $\mathbf{M}(k)_{\mathbb{Q}}$ by the so-called “virtual motives” produces an abelian, semi-simple category $\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}$ of pure motives for numerical equivalence. The objects of this category are now triples (X, p, m) , with $m \in \mathbb{Z}$. Effective motives are of course objects of this category and they are described by triples $(X, p, 0)$. The Lefschetz motive gains a new interpretation in this category as $\mathbf{L} = (\text{Spec}(k), \text{id}, -1)$. The Tate motive is defined by $\mathbf{L}^{-1} = (\text{Spec}(k), \text{id}, 1)$ and is therefore reminiscent of (in fact induces) the notion of Tate structure $\mathbb{Q}(1)$ in Hodge theory.

In the category $\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}$, the set of morphisms connecting two motives (X, p, m) and (Y, q, n) is defined by

$$\text{Hom}((X, p, m), (Y, q, n)) = q \circ A^{\dim X - m + n}(X \times Y) \circ p.$$

In particular, $\forall f = f^2 \in \text{End}((X, p, m))$, one defines the two motives

$$\text{Im}(f) = (X, p \circ f \circ p, m), \quad \text{Ker}(f) = (X, p - f, m).$$

These determine a canonical decomposition of any virtual motive as $\text{Im}(f) \oplus \text{Im}(1 - f) \xrightarrow{\sim} (X, p, m)$, where the direct sum of two motives as the above ones is defined by the formula

$$(X, p, m) \oplus (Y, q, m) = (X \amalg Y, p + q, m).$$

The general definition of the direct sum of two motives requires a bit more of formalism which escapes this short overview: we refer to *op.cit.* for the details.

The tensor structure

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \otimes q, m + n)$$

and the involution (*i.e.* auto-duality) which is defined, for X irreducible and $\dim X = d$ by the functor

$$\vee : (\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}})^{op} \rightarrow \mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}, \quad (X, p, m)^{\vee} = (X, p^t, d - m)$$

(the general case follows from this by applying additivity), determine the structure of a *rigid* \otimes -category on $\mathcal{M}_k(\mathcal{V}_k)_{\mathbb{Q}}$. Here, p^t denotes the transpose correspondence associated to p (*i.e.* the transpose of the graph). One finds, for example, that $\mathbf{L}^{\vee} = \mathbf{L}^{-1}$. In the particular case of the effective motive $h(X)$, with X irreducible and $\dim X = d$, this involution determines the notion of Poincaré duality

$$h(X)^{\vee} = h(X) \otimes \mathbf{L}^{\otimes(-d)}$$

that is an auto-duality which induces the Poincaré duality isomorphism in any classical cohomological theory.

2.3 Fundamental structures

A category of pure motives over a field k and with coefficients in a field K (of characteristic zero) is supposed to satisfy, to be satisfactory, several basic properties and to be endowed with a few fundamental structures. In the previous section we have described the historically-first example of a category of pure motives and

we have reviewed some of its basic properties (in that case $K = \mathbb{Q}$). One naturally wonders about the description of others categories of motives associated to finer (than the numerical) equivalence relations on algebraic cycles: namely the categories of motives for homological or rational or algebraic equivalence relations. However, if one seeks to work with a semi-simple category, the afore mentioned result of Jannsen tells us that the numerical equivalence is the only adequate relation. The semi-simplicity property is also attain if one assumes Grothendieck's Standard Conjectures. Following the report of Grothendieck in [20], these conjectures arose from the hope to understand (read prove) the conjectures of Weil on the zeta-function of an algebraic variety defined over a finite field. It was well-known to Grothendieck that the Standard Conjectures imply the Weil's Conjectures. These latter statements became a theorem in the early seventies (1974), only a few years later the time when Grothendieck stated the Standard Conjectures (1968-69). The proof by Deligne of the Weil's conjectures, however, does not make any use of the Standard Conjectures, these latter questions remain still unanswered at the present time. The moral lesson seems to be that geometric topology and the theory of algebraic cycles govern in many central aspects the foundations of algebraic geometry.

The Standard Conjectures of "Lefschetz type" and of "Hodge type" are stated in terms of algebraic cycles modulo homological equivalence (*cf.* [26]). They imply two further important conjectures. One of these states the equality of the homological and the numerical equivalence relations, the other one, of "Künneth type" claims that the Künneth decomposition of the diagonal in cohomology, can be described by means of (rational) algebraic cycles. It is generally accepted, nowadays, to refer to the full set of the four conjectures, when one quotes the Standard Conjectures. In view of their expected consequences, one is naturally led to study a category of pure *motives for homological equivalence*. In fact, there are several candidates for this category since the definition depends upon the choice of a Weil cohomological theory (*i.e.* Betti, étale, de-Rham, crystalline, etc) with coefficients in a field K of characteristic zero.

Let us fix a cohomological theory $X \mapsto H^*(X) = H^*(X, K)$ for algebraic varieties in the category \mathcal{V}_k . Then, the construction of the corresponding category of motives $\mathcal{M}_k^{hom}(\mathcal{V}_k)_K$ for homological equivalence is given following a procedure similar to that we have explained earlier on in this paper for the category of motives for numerical equivalence and with rational coefficients. The only difference is that now morphisms in the category $\mathcal{M}_k^{hom}(\mathcal{V}_k)_K$ are defined by means of algebraic correspondences modulo homological equivalence. At this point, one makes explicit use of the axiom "cycle map" that characterizes (together with finiteness, Poincaré duality, Künneth formula, cycle map, weak and strong Lefschetz theorems) any Weil cohomological theory (*cf.* [26]).

The set of algebraic morphisms connecting objects in the category \mathcal{V}_k is enlarged by including multi-valued maps $X \dashrightarrow Y$ that are defined as a K -linear combination of elements of the vector spaces

$$C^*(X \times Y) \otimes_{\mathbb{Z}} F / \sim_{hom},$$

where $F \subset K$ is a subfield. Two cycles $Z, W \in C^*(X \times Y) \otimes F$ are *homologically equivalent* $Z \sim_{hom} W$ if their image, by means of the cycle class map

$$\gamma : C^*(X \times Y) \otimes F \rightarrow H^*(X \times Y)$$

is the same. This leads naturally to the definition of a subvector-space $A_{hom}^*(X \times Y) \subset H^*(X \times Y)$ generated by the image of the cycle class map γ . These spaces define the correspondences in the category $\mathcal{M}_k^{hom}(\mathcal{V}_k)_K$. If X is purely d -dimensional, then

$$\text{Corr}^r(X, Y) := A_{hom}^{d+r}(X \times Y).$$

In general, if X decomposes into several connected components $X = \coprod_i X_i$, one sets $\text{Corr}^r(X, Y) = \oplus_i \text{Corr}^r(X_i, Y)$. In direct analogy to the construction of correspondences for numerical equivalence, the ring structure (“cap-product”) in cohomology determines a composition law ‘ \circ ’ among correspondences.

The category $\mathcal{M}_k^{hom}(\mathcal{V}_k)_K$ is then defined as follows: the objects are triples $M = (X, p, m)$, where $X \in \text{Obj}(\mathcal{V}_k)$, $m \in \mathbb{Z}$ and $p = p^2 \in \text{Corr}^0(X, X)$ is an idempotent. The collection of morphisms between two motives $M = (X, p, m)$, $N = (Y, q, n)$ is given by the set

$$\text{Hom}(M, N) = q \circ \text{Corr}^{n-m}(X, Y) \circ p.$$

This procedure determines a pseudo-abelian, K -linear tensor category. The tensor law is given by the formula

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \times q, m + n).$$

The commutativity and associativity constraints are induced by the obvious isomorphisms $X \times Y \xrightarrow{\sim} Y \times X$, $X \times (Y \times Z) \xrightarrow{\sim} (X \times Y) \times Z$. The unit object in the category is given by $\mathbf{1} = (\text{Spec}(k), \text{id}, 0)$. One shows that $\mathcal{M}_k^{hom}(\mathcal{V}_k)_K$ is a *rigid* category, as it is endowed with an auto-duality functor

$$\vee : \mathcal{M}_k^{hom}(\mathcal{V}_k)_K \rightarrow (\mathcal{M}_k^{hom}(\mathcal{V}_k)_K)^{op}.$$

For any object M , the functor $- \otimes M^\vee$ is left-adjoint to $- \otimes M$ and $M^\vee \otimes -$ is right-adjoint to $M \otimes -$. In the case of an irreducible variety X , the internal Hom is defined by the motive

$$\underline{\text{Hom}}((X, p, m), (Y, q, n)) = (X \times Y, p^t \times q, \dim(X) - m + n).$$

The Standard Conjecture of Künneth type (which is assumed from now on in this section) implies that the Künneth components of the diagonal $\pi_X^i \in \text{Corr}^0(X, X)$ determine a complete system of orthogonal, central idempotents. This important statement implies that the motive $h(X) \in \mathcal{M}_k^{hom}(\mathcal{V}_k)_K$ has the expected direct sum decomposition (unique)

$$h(X) = \bigoplus_{i=0}^{2 \dim X} h^i(X), \quad h^i(X) = \pi_X^i h(X).$$

The cohomology functor $X \mapsto H^*(X)$ factors through the projection $h(X) \rightarrow h^i(X)$. More in general, one shows that every motive $M = (X, p, m)$ gets this way

a \mathbb{Z} -grading structure by setting

$$(X, p, m)^r = (X, p \circ \pi^{r+2m}, r). \quad (2.8)$$

This grading is respected by all morphisms in the category and defines the structure of a *graduation by weights* on the objects. On a motive $M = (X, p, m) = ph(X) \otimes \mathbf{L}^{\otimes(-m)} = ph(X)(m)$ in the category, one sets

$$M = \bigoplus_i \mathrm{Gr}_i^w(M), \quad \mathrm{Gr}_i^w(M) = ph^{2m+i}(X)(m)$$

where $\mathrm{Gr}_i^w(M)$ is a pure motive of weight i . One finds for example, that $\mathbf{1}$ has weight zero, $\mathbf{L} = (\mathrm{Spec}(k), 1, -1)$ has weight 2 and that $\mathbf{L}^{-1} = (\mathrm{Spec}(k), 1, 1)$ has weight -2 . More in general, the motive $M = (X, p, m) = ph(X)(m)$ has weight $-2m$.

In order to achieve further important properties, one needs to modify the natural commutativity constraint $\psi = \bigoplus_{r,s} \psi^{r,s}, \psi^{r,s} : M^r \otimes N^s \xrightarrow{\sim} N^s \otimes M^r$, by defining

$$\psi_{new} = \bigoplus_{r,s} (-1)^{rs} \psi^{r,s}. \quad (2.9)$$

We shall denote by $\tilde{\mathcal{M}}_k^{hom}(\mathcal{V}_k)_K$ the category of motives for homological equivalence in which one has implemented the modification (2.9) on the tensor product structure.

An important structure on a category of pure motives (for homological equivalence) is given by assigning to an object $X \in \mathrm{Obj}(\mathcal{M}_k^{hom}(\mathcal{V}_k)_K)$ a *motivic cohomology* $H_{mot}^i(X)$. $H_{mot}^i(X)$ is a pure motive of weight i . This way, one views pure motives as a universal cohomological theory for algebraic varieties. The main property of the motivic cohomology is that it defines a universal realization of any given Weil cohomology theory H^* . Candidates for these motivic cohomology theories have been proposed by A. Beilinson [3], in terms of eigenspaces of Adams operations in algebraic K -theory *i.e.* $H_{mot}^{2j-n}(X, \mathbb{Q}(j)) = K_n(X)^{(j)}$ and by S. Bloch [5], in terms of higher Chow groups *i.e.* $H_{mot}^{2j-n}(X, \mathbb{Q}(j)) = CH^j(X, n) \otimes \mathbb{Q}$.

The assignment of a Weil cohomological theory with coefficients in a field K which contains an assigned field F is equivalent to the definition of an exact *realization* \otimes -functor of $\mathcal{M}_k^{hom}(\mathcal{V}_k)_K$ in the category of K -vector spaces

$$r_{H^*} : \tilde{\mathcal{M}}_k^{hom}(\mathcal{V}_k)_K \rightarrow \mathrm{Vect}_K, \quad r_{H^*}(H_{mot}^i(X)) \simeq H^i(X). \quad (2.10)$$

In particular, one obtains the realization $r_{H^*}(\mathbf{L}^{-1}) = H^2(\mathbf{P}^1)$ which defines the notion of the *Tate twist* in cohomology. More precisely:

– in étale cohomology: $H^2(\mathbf{P}^1) = \mathbb{Q}_\ell(-1)$, where $\mathbb{Q}_\ell(1) := \varprojlim_m \mu_{\ell^m}$ is a \mathbb{Q}_ℓ -vector space of dimension one endowed with the cyclotomic action of the absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$. The “twist” (or torsion) (r) in étale cohomology corresponds to the torsion in Galois theory defined by the r -th power of the cyclotomic character (Tate twist)

– in de-Rham theory: $H_{DR}^2(\mathbf{P}^1) = k$, with the Hodge filtration defined by $F^{\leq 0} = 0, F^{> 0} = k$. Here, the effect of the torsion (r) is that of shifting the Hodge filtration of $-r$ -steps (to the right)

– in Betti theory: $H^2(\mathbf{P}^1) = \mathbb{Q}(-1) := (2\pi i)^{-1}\mathbb{Q}$. The bi-graduation on $H^2(\mathbf{P}^1) \otimes \mathbb{C} \simeq \mathbb{C}$ is purely of type $(1, 1)$. The torsion (r) is here identified with the composite of a homothety given by a multiplication by $(2\pi i)^{-r}$ followed by a shifting by $(-r, -r)$ of the Hodge bi-graduation.

Using the structure of rigid tensor-category one introduces the notion of *rank* associated to a motive $M = (X, p, m)$ in $\mathcal{M}_k^{hom}(\mathcal{V}_k)_K$. The rank of M is defined as the trace of id_M *i.e.* the trace of the morphism $\epsilon \circ \psi_{new} \circ \eta \in \text{End}(\mathbf{1})$, where

$$\epsilon : M \otimes M^\vee \rightarrow \mathbf{1}, \quad \eta : \mathbf{1} \rightarrow M^\vee \otimes M$$

are *resp.* the evaluation and co-evaluation morphisms satisfying $\epsilon \otimes \text{id}_M \circ \text{id}_M \otimes \eta = \text{id}_M$, $\text{id}_{M^\vee} \otimes \epsilon \circ \eta \otimes \text{id}_{M^\vee} = \text{id}_{M^\vee}$. In general, one sets

$$\text{rk}(X, p, m) = \sum_{i \geq 0} \dim p H^i(X) \geq 0.$$

Under the assumption of the Standard Conjectures (more precisely under the assumption that homological and numerical equivalence relations coincide) and that $\text{End}(\mathbf{1}) = F$ ($\text{char}(F) = 0$), the tannakian formalism invented by Grothendieck and developed by Saavedra [33], and Deligne [16] implies that the abelian, rigid, semi-simple tensor category $\tilde{\mathcal{M}}_k^{hom}(\mathcal{V}_k)_K$ is endowed with an exact, faithful \otimes -fibre functor to the category of graded K -vector spaces

$$\omega : \tilde{\mathcal{M}}_k^{hom}(\mathcal{V}_k)_K \rightarrow \text{VectGr}_K, \quad \omega(H_{mot}^*(X)) = H^*(X) \quad (2.11)$$

which is compatible with the realization functor. This formalism defines a *tannakian* (*neutral* if $K = F$) structure on the category of motives. One then introduces the *tannakian group*

$$G = \underline{\text{Aut}}^\otimes(\omega)$$

as a K -scheme in affine groups. Through the tannakian formalism one shows that the fibre functor ω realizes an equivalence of rigid tensor categories

$$\omega : \tilde{\mathcal{M}}_k^{hom}(\mathcal{V}_k)_K \xrightarrow{\sim} \text{Rep}_F(G),$$

where $\text{Rep}_F(G)$ denotes the rigid tensor category of finite dimensional, F representations of the tannakian group G . This way, one establishes a quite useful dictionary between categorical \otimes -properties and properties of the associated groups. Because we have assumed all along the Standard Conjectures, the semi-simplicity of the category $\tilde{\mathcal{M}}_k^{hom}(\mathcal{V}_k)_K$ implies that G is an algebraic, pro-reductive group, that is G is the projective limit of reductive F -algebraic groups.

The tannakian theory is a linear analog of the theory of finite, étale coverings of a given connected scheme. This theory was developed by Grothendieck in SGA1 (theory of the pro-finite π_1). For this reason the group G is usually referred to as the *motivic Galois group* associated to \mathcal{V}_k and H^* . In the case of algebraic varieties of dimension zero (*i.e.* for Artin motives) the tannakian group G is nothing but the (absolute) Galois group $\text{Gal}(\bar{k}/k)$.

In any reasonable cohomological theory the functors $X \mapsto H^*(X)$ are deduced by applying standard methods of homological algebra to the related derived functors $X \mapsto R\Gamma(X)$ which associate to an object in \mathcal{V}_k a bounded complex of k -vector

spaces, in a suitable triangulated category $\mathcal{D}(k)$ of complexes of modules over k , whose heart is the category of motives. This is the definition of cohomology as

$$H^i(X) = H^i R\Gamma(X).$$

Under the assumption that the functors $R\Gamma$ are realizations of corresponding motivic functors *i.e.* $R\Gamma = r_{H^*} R\Gamma_{mot}$, one expects the existence of a (non-canonical) isomorphism in $\mathcal{D}(k)$

$$R\Gamma_{mot}(X) \simeq \oplus_i H_{mot}^i(X)[-i]. \quad (2.12)$$

Moreover, the introduction of the motivic derived functors $R\Gamma_{mot}$ suggests the definition of the following groups of *absolute cohomology*

$$H_{abs}^i(X) = \text{Hom}_{\mathcal{D}(k)}(\mathbf{1}, R\Gamma_{mot}(X)[i]).$$

For a general motive $M = (X, p, m)$, one defines

$$H_{abs}^i(M) = \text{Hom}_{\mathcal{D}(k)}(\mathbf{1}, M[i]) = \text{Ext}^i(\mathbf{1}, M). \quad (2.13)$$

The motives $H_{mot}^i(X)$ and the groups of absolute motivic cohomology are related by a spectral sequence

$$E_2^{p,q} = H_{abs}^p(H_{mot}^q(X)) \Rightarrow H_{abs}^{p+q}(X).$$

2.4 Examples of pure motives

The first interesting examples of pure motives arise by considering the category \mathcal{V}_k^o of étale, finite k -schemes. An object in this category is a scheme $X = \text{Spec}(k')$, where k' is a commutative k -algebra of finite dimension which satisfies the following properties. Let \bar{k} denote a fixed separable closure of k

1. $k' \otimes \bar{k} \simeq \bar{k}^{[k':k]}$
2. $k' \simeq \prod k_\alpha$, for k_α/k finite, separable field extensions
3. $|X(\bar{k})| = [k' : k]$.

The corresponding rigid, tensor-category of motives with coefficients in a field K is usually referred to as the category of *Artin motives*: $\mathcal{CV}^o(k)_K$.

The definition of this category is independent of the choice of the equivalence relation on cycles as the objects of \mathcal{V}_k^o are smooth, projective k -varieties of dimension zero. One also sees that passing from \mathcal{V}_k^o to $\mathcal{CV}^o(k)_K$ requires adding new objects in order to attain the property that the category of motives is abelian. One can verify this already for $k = K = \mathbb{Q}$, by considering the real quadratic extension $k' = \mathbb{Q}(\sqrt{2})$ and the one-dimensional non-trivial representation of $G_{\mathbb{Q}} = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that factors through the character of order two of $\text{Gal}(k'/\mathbb{Q})$. This representation does not correspond to any object in $\mathcal{V}_{\mathbb{Q}}^o$, but can be obtained as the image of the projector $p = \frac{1}{2}(1 - \sigma)$, where σ is the generator of $\text{Gal}(k'/\mathbb{Q})$. Therefore, $\text{image}(p) \in \text{Obj}(\mathcal{CV}^o(\mathbb{Q})_{\mathbb{Q}})$ is a new object.

The category of Artin motives is a semi-simple, K -linear, monoidal \otimes -category. When $\text{char}(K) = 0$, the commutative diagram of functors

$$\begin{array}{ccc} \mathcal{V}_k^o & \xrightarrow[\sim]{GG} & \{\text{sets with } \text{Gal}(\bar{k}/k)\text{-continuous action}\} \\ \downarrow h & & \downarrow l \\ \mathcal{CV}^o(k)_K & \xrightarrow[\sim]{(*)} & \{\text{finite dim. } K\text{-v.spaces with linear } \text{Gal}(\bar{k}/k)\text{-continuous action}\} \end{array}$$

where l is the contravariant functor of linearization

$$S \mapsto K^S, \quad (g(f))(s) = f(g^{-1}(s)), \quad \forall g \in \text{Gal}(\bar{k}/k)$$

determines a linearization of the Galois-Grothendieck correspondence (GG) by means of the equivalence of categories (*). This is provided by the fiber functor

$$\omega : X \rightarrow H^0(X_{\bar{k}}, K) = K^{X(\bar{k})}$$

and by applying the tannakian formalism. It follows that $\mathcal{CV}^o(k)_K$ is \otimes -equivalent to the category $\text{Rep}_K \text{Gal}(\bar{k}/k)$ of representations of the absolute Galois group $G_k = \text{Gal}(\bar{k}/k)$.

These results were the departing point for Grothendieck's speculations on the definition of higher dimensional Galois theories (*i.e.* Galois theories associated to system of polynomials in several variables) and for the definition of the corresponding motivic Galois groups.

3 Endomotives: an overview

The notion of an *endomotive* in noncommutative geometry (*cf.* [7]) is the natural generalization of the classical concept of an Artin motive for the noncommutative spaces which are defined by semigroup actions on projective limits of zero-dimensional algebraic varieties, endowed with an action of the absolute Galois group. This notion applies quite naturally for instance, to the study of several examples of quantum statistical dynamical systems whose time evolution describes important number-theoretic properties of a given field k (*cf. op.cit.*, [14]).

There are two distinct definitions of an endomotive: one speaks of algebraic or analytic endomotives depending upon the context and the applications.

When k is a number field, there is a functor connecting the two related categories. Moreover, the abelian category of Artin motives embeds naturally as a full subcategory in the category of algebraic endomotives (*cf.* Theorem 5.3) and this result motivates the statement that the theory of endomotives defines a natural generalization of the classical theory of (zero-dimensional) Artin motives.

In noncommutative geometry, where the properties of a space (frequently highly singular from a classical viewpoint) are analyzed in terms of the properties of the associated noncommutative algebra and its (space of) irreducible representations, it is quite natural to look for a suitable abelian category which enlarges the original,

non-additive category of algebras and in which one may also apply the standard techniques of homological algebra. Likewise in the construction of a theory of motives, one seeks to work within a triangulated category endowed with several structures. These include for instance, the definition of (noncommutative) motivic objects playing the role of motivic cohomology (*cf.* (2.12)), the construction of a universal (co)homological theory representing in this context the absolute motivic cohomology (*cf.* (2.13)) and possibly also the set-up of a noncommutative tannakian formalism to motivate in rigorous mathematical terms the presence of certain universal groups of symmetries associated to renormalizable quantum field theories (*cf. e.g.* [12]).

A way to attack these problems is that of enlarging the original category of algebras and morphisms by introducing a “derived” category of modules enriched with a suitable notion of correspondences connecting the objects that should also account for the structure of Morita equivalence which represents the noncommutative generalization of the notion of isomorphism for commutative algebras.

3.1 The abelian category of cyclic modules

The sought for enlargement of the category Alg_k of (unital) k -algebras and (unital) algebra homomorphisms is defined by introducing a new category Λ_k of cyclic $k(\Lambda)$ -modules. The objects of this category are modules over the *cyclic category* Λ . This latter has the same objects as the simplicial category Δ (Λ contains Δ as sub-category). We recall that an object in Δ is a totally ordered set

$$[n] = \{0 < 1 < \dots < n\}$$

for each $n \in \mathbb{N}$, and a morphism

$$f : [n] \rightarrow [m]$$

is described by an order-preserving map of sets $f : \{0, 1, \dots, n\} \rightarrow \{0, 1, \dots, m\}$. The set of morphisms in Δ is generated by faces $\delta_i : [n-1] \rightarrow [n]$ (the injection that misses i) and degeneracies $\sigma_j : [n+1] \rightarrow [n]$ (the surjection which identifies j with $j+1$) which satisfy several standard simplicial identities (*cf. e.g.* [9]). The set of morphisms in Λ is enriched by introducing a new collection of morphisms: the *cyclic morphisms*. For each $n \in \mathbb{N}$, one sets

$$\tau_n : [n] \rightarrow [n]$$

fulfilling the relations

$$\begin{aligned} \tau_n \delta_i &= \delta_{i-1} \tau_{n-1} & 1 \leq i \leq n, & \quad \tau_n \delta_0 = \delta_n \\ \tau_n \sigma_i &= \sigma_{i-1} \tau_{n+1} & 1 \leq i \leq n, & \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \\ \tau_n^{n+1} &= 1_n. \end{aligned} \tag{3.1}$$

The objects of the category Λ_k are k -modules over Λ (*i.e.* $k(\Lambda)$ -modules). In categorical language this means functors

$$\Lambda^{op} \rightarrow Mod_k$$

($Mod_k =$ category of k -modules). Morphisms of $k(\Lambda)$ -modules are therefore natural transformations between the corresponding functors.

It is evident that Λ_k is an abelian category, because of the interpretation of a morphism in Λ_k as a collection of k -linear maps of k -modules $A_n \rightarrow B_n$ ($A_n, B_n \in \text{Obj}(Mod_k)$) compatible with faces, degeneracies and cyclic operators. Kernels and cokernels of these morphisms define objects of the category Λ_k , since their definition is given point-wise.

To an algebra \mathcal{A} over a field k , one associates the $k(\Lambda)$ -module \mathcal{A}^\natural . For each $n \geq 0$ one sets

$$\mathcal{A}_n^\natural = \underbrace{\mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{(n+1)\text{-times}}.$$

The *cyclic morphisms* on \mathcal{A}^\natural correspond to the cyclic permutations of the tensors, while the face and the degeneracy maps correspond to the algebra product of consecutive tensors and the insertion of the unit. This construction determines a functor

$$\natural : Alg_k \rightarrow \Lambda_k$$

Traces $\varphi : \mathcal{A} \rightarrow k$ give rise naturally to morphisms

$$\varphi^\natural : \mathcal{A}^\natural \rightarrow k^\natural, \quad \varphi^\natural(a_0 \otimes \cdots \otimes a_n) = \varphi(a_0 \cdots a_n)$$

in Λ_k . The main result of this construction is the following canonical description of the cyclic cohomology of an algebra \mathcal{A} over a field k as the derived functor of the functor which assigns to a $k(\Lambda)$ -module its space of traces

$$HC^n(\mathcal{A}) = \text{Ext}^n(\mathcal{A}^\natural, k^\natural) \tag{3.2}$$

(*cf.* [9],[29]). This formula is the analog of (2.13), that describes the absolute motivic cohomology group of a classical motive as an Ext-group computed in a triangulated category of motives $\mathcal{D}(k)$. In the present context, on the other hand, the derived groups Ext^n are taken in the abelian category of Λ_k -modules.

The description of the cyclic cohomology as a derived functor in the cyclic category determines a useful procedure to embed the nonadditive category of algebras and algebra homomorphisms in the “derived” abelian category of $k(\Lambda)$ -modules. This construction provides a natural framework for the definition of the objects of a category of noncommutative motives.

Likewise in the construction of the category of motives, one is faced with the problem of finding the “motivated maps” connecting cyclic modules. The natural strategy is that of enlarging the collection of cyclic morphisms which are functorially induced by homomorphisms between (noncommutative) algebras, by implementing an adequate definition of (noncommutative) correspondences. The notion of an algebraic correspondence in algebraic geometry, as a multi-valued map defined by an algebraic cycle modulo a suitable equivalence relation, has here

an analog with the notion of Kasparov’s bimodule and the associated class in KK -theory (cf. [25]). Likewise in classical motive theory, one may prefer to work with (compare) several versions of correspondences. One may decide to retain the full information supplied by a group action on a given algebra (*i.e.* a noncommutative space) rather than partially losing this information by moding out with the equivalence relation (homotopy in KK -theory).

3.2 Bimodules and KK -theory

There is a natural way to associate a cyclic morphism to a (virtual) correspondence and hence to a class in KK -theory. Starting with the category of separable C^* -algebras and $*$ -homomorphisms, one enlarges the collection of morphisms connecting two unital algebras \mathcal{A} and \mathcal{B} , by including correspondences defined by elements of Kasparov’s bivariant K -theory

$$\text{Hom}(\mathcal{A}, \mathcal{B}) = KK(\mathcal{A}, \mathcal{B})$$

([25], cf. also §8 and §9.22 of [4]). More precisely, correspondences are defined by Kasparov’s bimodules, that means by triples

$$\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B}) = (E, \phi, F)$$

which satisfy the following conditions:

- E is a countably generated Hilbert module over \mathcal{B}
- ϕ is a $*$ -homomorphism of \mathcal{A} to bounded linear operators on E (*i.e.* ϕ gives E the structure of an \mathcal{A} - \mathcal{B} bimodule)
- F is a bounded linear operator on E such that the operators $[F, \phi(a)]$, $(F^2 - 1)\phi(a)$, and $(F^* - F)\phi(a)$ are compact for all $a \in \mathcal{A}$.

A Hilbert module E over \mathcal{B} is a right \mathcal{B} -module with a *positive*, \mathcal{B} -valued inner product which satisfies $\langle x, yb \rangle = \langle x, y \rangle b$, $\forall x, y \in E$ and $\forall b \in \mathcal{B}$, and with respect to which E is complete (*i.e.* complete in the norm $\|x\| = \sqrt{\|\langle x, x \rangle\|}$).

Notice that Kasparov bimodules are Morita-type of correspondences. They generalize $*$ -homomorphisms of C^* -algebras since the latter ones may be re-interpreted as Kasparov bimodules of the form $(\mathcal{B}, \phi, 0)$.

Given a Kasparov’s bimodule $\mathcal{E} = \mathcal{E}(\mathcal{A}, \mathcal{B})$, that is a \mathcal{A} - \mathcal{B} Hilbert bimodule E as defined above, one associates, under the assumption that E is a projective \mathcal{B} -module of finite type (cf. [7], Lemma 2.1), a cyclic morphism

$$\mathcal{E}^\natural \in \text{Hom}(\mathcal{A}^\natural, \mathcal{B}^\natural).$$

This result allows one to define an enlargement of the collection of cyclic morphisms in the category Λ_k of $k(\Lambda)$ -modules, by considering Kasparov’s projective bimodules of finite type, as correspondences.

One then implements the *homotopy equivalence relation* on the collection of Kasparov’s bimodules. Two Kasparov’s modules are said to be homotopy equivalent $(E_0, \phi_0, F_0) \sim_h (E_1, \phi_1, F_1)$ if there is an element

$$(E, \phi, F) \in \mathcal{E}(\mathcal{A}, I\mathcal{B}), \quad I\mathcal{B} = \{f : [0, 1] \rightarrow \mathcal{B} \mid f \text{ continuous}\}$$

which performs a unitary homotopy deformation between the two modules. This means that $(E \hat{\otimes}_{f_i} \mathcal{B}, f_i \circ \phi, f_i(F))$ is unitarily equivalent to (E_i, ϕ_i, F_i) or equivalently re-phrased, that there is a unitary in bounded operators from $E \hat{\otimes}_{f_i} \mathcal{B}$ to E_i intertwining the morphisms $f_i \circ \phi$ and ϕ_i and the operators $f_i(F)$ and F_i . Here $f_i : I\mathcal{B} \rightarrow \mathcal{B}$ is the evaluation at the endpoints.

There is a binary operation on the set of all Kasparov \mathcal{A} - \mathcal{B} bimodules, given by the direct sum. By definition, the group of Kasparov's bivariant K -theory is the set of homotopy equivalence classes $c(\mathcal{E}(\mathcal{A}, \mathcal{B})) \in KK(\mathcal{A}, \mathcal{B})$ of Kasparov's modules $\mathcal{E}(\mathcal{A}, \mathcal{B})$. This set has a natural structure of abelian group with addition induced by direct sum.

This bivariant version of K -theory is richer than both K -theory and K -homology, as it carries an intersection product. There is a natural bi-linear, associative composition (intersection) product

$$\otimes_{\mathcal{B}} : KK(\mathcal{A}, \mathcal{B}) \times KK(\mathcal{B}, \mathcal{C}) \rightarrow KK(\mathcal{A}, \mathcal{C})$$

for all \mathcal{A}, \mathcal{B} and \mathcal{C} separable C^* -algebras. This product is compatible with composition of morphisms of C^* -algebras.

KK -theory is also endowed with a bi-linear, associative exterior product

$$\otimes : KK(\mathcal{A}, \mathcal{B}) \otimes KK(\mathcal{C}, \mathcal{D}) \rightarrow KK(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{D}),$$

which is defined in terms on the composition product by

$$c_1 \otimes c_2 = (c_1 \otimes 1_{\mathcal{C}}) \otimes_{\mathcal{B} \otimes \mathcal{C}} (c_2 \otimes 1_{\mathcal{B}}).$$

A slightly different formulation of KK -theory, which simplifies the definition of this external tensor product is obtained by replacing in the data (E, ϕ, F) the operator F by an *unbounded*, regular self-adjoint operator D . The corresponding F is then given by $D(1 + D^2)^{-1/2}$ (*cf.* [1]).

The above construction which produces an enlargement of the category of separable C^* -algebras by introducing correspondences as morphisms determines an additive, although non abelian category \mathcal{KK} (*cf.* [4] §9.22.1). This category is also known to have a triangulated structure (*cf.* [32]) and this result is in agreement with the construction of the triangulated category $\mathcal{D}(k)$ in motives theory, whose heart is expected to be the category of (mixed) motives (*cf.* section 2.3 and [17]). A more refined analysis based on the analogy with the construction of a category of motives suggests that one should probably perform a further enlargement by passing to the pseudo-abelian envelope of \mathcal{KK} , that is by formally including among the objects also ranges of idempotents in KK -theory.

In section 5 we will review the category of analytic endomotives where maps are given in terms of étale correspondences described by spaces \mathcal{Z} arising from locally compact étale groupoids $\mathcal{G} = \mathcal{G}(X_\alpha, S, \mu)$ associated to zero-dimensional, singular quotient spaces $X(\bar{k})/S$ with associated C^* -algebras $C^*(\mathcal{G})$. In view of what we have said in this section, it would be also possible to define a category where morphisms are given by classes $c(\mathcal{Z}) \in KK(C^*(\mathcal{G}), C^*(\mathcal{G}'))$ which describe sets of equivalent triples (E, ϕ, F) , where (E, ϕ) is given in terms of a bimodule $\mathcal{M}_{\mathcal{Z}}$ with the trivial grading $\gamma = 1$ and the zero endomorphism $F = 0$. The definition of

the category of analytic endomotives is more refined because the definition of the maps in this category does not require to divide by homotopy equivalence.

The comparison between correspondences for motives given by algebraic cycles and correspondences for noncommutative spaces given by bimodules (or elements in KK -theory) is particularly easy in the zero-dimensional case because the equivalence relations play no role. Of course, it would be quite interesting to investigate the higher dimensional cases, in view of a unified framework for motives and noncommutative spaces which is suggested for example, by the recent results on the Lefschetz trace formula for archimedean local factors of L -functions of motives (*cf.* [7], Section 7).

A way to attack this problem is by comparing the notion of a correspondence given by an algebraic cycle with the notion of a geometric correspondence used in topology (*cf.* [2], [13]). For example, it is easy to see that the definition of an algebraic correspondence can be reformulated as a particular case of the topological (geometrical) correspondence and it is also known that one may associate to the latter a class in KK -theory. In the following two sections, we shall review and comment on these ideas.

3.3 Geometric correspondences

In geometric topology, given two smooth manifolds X and Y (it is enough to assume that X is a locally compact parameter space), a *topological (geometric) correspondence* is given by the datum

$$X \xleftarrow{f_X} (Z, E) \xrightarrow{g_Y} Y$$

where:

- Z is a smooth manifold
- E is a complex vector bundle over Z
- $f_X : Z \rightarrow X$ and $g_Y : Z \rightarrow Y$ are continuous maps, with f_X proper and g_Y K -oriented (orientation in K -homology).

Unlike in the definition of an algebraic correspondence (*cf.* section 2.3) one does not require that Z is a subset of the cartesian product $X \times Y$. This flexibility is balanced by the implementation of the extra piece of datum given by the vector bundle E . To any such correspondence (Z, E, f_X, g_Y) one associates a class in Kasparov's K -theory

$$c(Z, E, f_X, g_Y) = (f_X)_*((E) \otimes_Z (g_Y)!) \in KK(X, Y). \quad (3.3)$$

(E) denotes the class of E in $KK(Z, Z)$ and $(g_Y)!$ is the element in KK -theory which fulfills the Grothendieck Riemann-Roch formula.

We recall that given two smooth manifolds X_1 and X_2 and a continuous oriented map $f : X_1 \rightarrow X_2$, the element $f! \in KK(X_1, X_2)$ determines the Grothendieck Riemann-Roch formula

$$\text{ch}(F \otimes f!) = f!(\text{Td}(f) \cup \text{ch}(F)), \quad (3.4)$$

for all $F \in K^*(X_1)$, with $\text{Td}(f)$ the Todd genus

$$\text{Td}(f) = \text{Td}(TX_1)/\text{Td}(f^*TX_2). \quad (3.5)$$

The composition of two correspondences (Z_1, E_1, f_X, g_Y) and (Z_2, E_2, f_Y, g_W) is given by taking the fibered product $Z = Z_1 \times_Y Z_2$ and the bundle $E = \pi_1^*E_1 \times \pi_2^*E_2$, with $\pi_i : Z \rightarrow Z_i$ the projections. This determines the composite correspondence (Z, E, f_X, g_W) . In fact, one also needs to assume a *transversality* condition on the maps g_Y and f_Y in order to ensure that the fibered product Z is a smooth manifold. The homotopy invariance of both $g_Y!$ and $(f_X)_*$ show however that the assumption of transversality is ‘generically’ satisfied.

Theorem 3.2 of [13] shows that Kasparov product in KK -theory ‘ \otimes ’ agrees with the composition of correspondences, namely

$$c(Z_1, E_1, f_X, g_Y) \otimes_Y c(Z_2, E_2, f_Y, g_W) = c(Z, E, f_X, g_W) \in KK(X, W). \quad (3.6)$$

3.4 Algebraic correspondences and K -theory

In algebraic geometry, the notion of correspondence that comes closest to the definition of a geometric correspondence (as an element in KK -theory) is obtained by considering classes of algebraic cycles in algebraic K -theory (*cf.* [30]).

Given two smooth and projective algebraic varieties X and Y , we denote by p_X and p_Y the projections of $X \times Y$ onto X and Y respectively and we assume that they are *proper*. Let $Z \in C^*(X \times Y)$ be an algebraic cycle. For simplicity, we shall assume that Z is irreducible (the general case follows by linearity). We denote by $f_X = p_X|_Z$ and $g_Y = p_Y|_Z$ the restrictions of p_X and p_Y to Z .

To the irreducible subvariety $T \xrightarrow{i} Y$ one naturally associates the coherent \mathcal{O}_Y -module $i_*\mathcal{O}_T$. For simplicity of notation we write it as \mathcal{O}_T . We use a similar notation for the coherent sheaf \mathcal{O}_Z , associated to the irreducible subvariety $Z \hookrightarrow X \times Y$. Then, the sheaf pullback

$$p_Y^*\mathcal{O}_T = p_Y^{-1}\mathcal{O}_T \otimes_{p_Y^{-1}\mathcal{O}_Y} \mathcal{O}_{X \times Y} \quad (3.7)$$

has a natural structure of $\mathcal{O}_{X \times Y}$ -module. The map on sheaves that corresponds to the cap product by Z on cocycles is given by

$$Z : \mathcal{O}_T \mapsto (p_X)_* (p_Y^*\mathcal{O}_T \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_Z). \quad (3.8)$$

Since p_X is proper, the resulting sheaf is coherent. Using (3.7), we can write equivalently

$$Z : \mathcal{O}_T \mapsto (p_X)_* \left(p_Y^{-1}\mathcal{O}_T \otimes_{p_Y^{-1}\mathcal{O}_Y} \mathcal{O}_Z \right). \quad (3.9)$$

We recall that the functor $f_!$ is the right adjoint to f^* (*i.e.* $f^*f_! = id$) and that $f_!$ satisfies the Grothendieck Riemann–Roch formula

$$\text{ch}(f_!(F)) = f_!(\text{Td}(f) \cup \text{ch}(F)). \quad (3.10)$$

Using this result, we can equally compute the intersection product of (3.8) by first computing

$$\mathcal{O}_T \otimes_{\mathcal{O}_Y} (p_Y)_! \mathcal{O}_Z \quad (3.11)$$

and then applying p_Y^* . Using (3.10) and (3.4) we know that we can replace (3.11) by $\mathcal{O}_T \otimes_{\mathcal{O}_Y} (\mathcal{O}_Z \otimes (p_Y)_!)$ with the same effect in K -theory.

Thus, to a correspondence in the sense of (3.8) that is defined by the image in K -theory of an algebraic cycle $Z \in C^*(X \times Y)$ we associate the geometric class

$$\mathcal{F}(Z) = c(Z, E, f_X, g_Y) \in KK(X, Y),$$

with $f_X = p_X|_Z$, $g_Y = p_Y|_Z$ and with the bundle $E = \mathcal{O}_Z$.

The composition of correspondences is given in terms of the intersection product of the associated cycles. Given three smooth projective varieties X , Y and W and (virtual) correspondences $U = \sum a_i Z_i \in C^*(X \times Y)$ and $V = \sum c_j Z'_j \in C^*(Y \times W)$, with $Z_i \subset X \times Y$ and $Z'_j \subset Y \times W$ closed reduced irreducible subschemes, one defines

$$U \circ V = (\pi_{13})_* ((\pi_{12})^* U \bullet (\pi_{23})^* V). \quad (3.12)$$

$\pi_{12} : X \times Y \times W \rightarrow X \times Y$, $\pi_{23} : X \times Y \times W \rightarrow Y \times W$, and $\pi_{13} : X \times Y \times W \rightarrow X \times W$ denote, as usual, the projection maps.

Under the assumption of ‘general position’ which is the algebraic analog of the transversality requirement in topology, we obtain the following result

Proposition 3.1 (cf. [7] Proposition 6.1). *Suppose given three smooth projective varieties X , Y , and W and algebraic correspondences U given by $Z_1 \subset X \times Y$ and V given by $Z_2 \subset Y \times W$. Assume that $(\pi_{12})^* Z_1$ and $(\pi_{23})^* Z_2$ are in general position in $X \times Y \times W$. Then assigning to a cycle Z the topological correspondence $\mathcal{F}(Z) = (Z, E, f_X, g_Y)$ satisfies*

$$\mathcal{F}(Z_1 \circ Z_2) = \mathcal{F}(Z_1) \otimes_Y \mathcal{F}(Z_2), \quad (3.13)$$

where $Z_1 \circ Z_2$ is the product of algebraic cycles and $\mathcal{F}(Z_1) \otimes_Y \mathcal{F}(Z_2)$ is the Kasparov product of the topological correspondences.

Notice that, while in the topological (smooth) setting transversality can always be achieved by a small deformation (cf. §III, [13]), in the algebro-geometric framework one needs to modify the above construction if the cycles are not in general position. In this case the formula

$$[\mathcal{O}_{T_1}] \otimes [\mathcal{O}_{T_2}] = [\mathcal{O}_{T_1 \circ T_2}]$$

which describes the product in K -theory in terms of the intersection product of algebraic cycles must be modified by implementing Tor-classes and one works with a product defined by the formula ([30], Theorem 2.7)

$$[\mathcal{O}_{T_1}] \otimes [\mathcal{O}_{T_2}] = \sum_{i=0}^n (-1)^i \left[\text{Tor}_i^{\mathcal{O}_X} (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}) \right]. \quad (3.14)$$

4 Algebraic endomotives

To define the category of *algebraic endomotives* one replaces the category \mathcal{V}_k^o of reduced, finite-dimensional commutative algebras (and algebras homomorphisms) over a field k by the category of noncommutative algebras (and algebras homomorphisms) of the form

$$\mathcal{A}_k = A \rtimes S.$$

A denotes a unital algebra which is an inductive limit of commutative algebras $A_\alpha \in \text{Obj}(\mathcal{V}_k^o)$. S is a unital, *abelian semigroup* of algebra endomorphisms

$$\rho : A \rightarrow A.$$

Moreover, one imposes the condition that for $\rho \in S$, $e = \rho(1) \in A$ is an *idempotent* of the algebra and that ρ is an isomorphism of A with the compressed algebra eAe .

The crossed product algebra \mathcal{A}_k is defined by formally adjoining to A new generators U_ρ and U_ρ^* , for $\rho \in S$, satisfying the algebraic rules

$$\begin{aligned} U_\rho^* U_\rho &= 1, & U_\rho U_\rho^* &= \rho(1), & \forall \rho \in S, \\ U_{\rho_1 \rho_2} &= U_{\rho_1} U_{\rho_2}, & U_{\rho_2 \rho_1}^* &= U_{\rho_1}^* U_{\rho_2}^*, & \forall \rho_j \in S, \\ U_\rho x &= \rho(x) U_\rho, & x U_\rho^* &= U_\rho^* \rho(x), & \forall \rho \in S, \forall x \in A. \end{aligned} \quad (4.1)$$

Since S is abelian, these rules suffice to show that \mathcal{A}_k is the linear span of the monomials $U_{\rho_1}^* a U_{\rho_2}$, for $a \in A$ and $\rho_j \in S$.

Because $A = \varinjlim_\alpha A_\alpha$, with A_α reduced, finite-dimensional commutative algebras

over k , the construction of \mathcal{A}_k is in fact determined by assigning a *projective system* $\{X_\alpha\}_{\alpha \in I}$ of varieties in \mathcal{V}_k^o (I is a countable indexing set), with $\xi_{\beta, \alpha} : X_\beta \rightarrow X_\alpha$ morphisms in \mathcal{V}_k^o and with a suitably defined action of S . Here, we have implicitly used the equivalence between the category of finite dimensional commutative k -algebras and the category of affine algebraic varieties over k .

The graphs $\Gamma_{\xi_{\beta, \alpha}}$ of the connecting morphisms of the system define $G_k = \text{Gal}(\bar{k}/k)$ -invariant subsets of $X_\beta(\bar{k}) \times X_\alpha(\bar{k})$ which in turn describe $\xi_{\beta, \alpha}$ as algebraic correspondences. We denote by

$$X = \varprojlim_\alpha X_\alpha, \quad \xi_\alpha : X \rightarrow X_\alpha$$

the associated *pro-variety*. The compressed algebra eAe associated to the idempotent $e = \rho(1)$ determines a subvariety $X^e \subset X$ which is in fact isomorphic to X , via the induced morphism $\tilde{\rho} : X \rightarrow X^e$.

The noncommutative space defined by \mathcal{A}_k is the quotient of $X(\bar{k})$ by the action of S , *i.e.* of the action of the $\tilde{\rho}$'s.

The Galois group G_k acts on $X(\bar{k})$ by composition. By identifying the elements of $X(\bar{k})$ with characters, *i.e.* with k -algebra homomorphisms $\chi : A \rightarrow \bar{k}$, we write the action of G_k on A as

$$\alpha(\chi) = \alpha \circ \chi : A \rightarrow \bar{k}, \quad \forall \alpha \in G_k, \forall \chi \in X(\bar{k}). \quad (4.2)$$

This action commutes with the maps $\tilde{\rho}$, *i.e.* $(\alpha \circ \chi) \circ \rho = \alpha \circ (\chi \circ \rho)$. Thus the whole construction of the system (X_α, S) is G_k -equivariant. This fact does not mean however, that G_k acts by automorphisms on \mathcal{A}_k !

Moreover, notice that the algebraic construction of the crossed-product algebra \mathcal{A}_k endowed with the actions of G_k and S on $X(\bar{k})$ makes sense also when $\text{char}(k) > 0$. When $\text{char}(k) = 0$, one defines the set of correspondences $M(\mathcal{A}_k, \mathcal{B}_k)$ by using the notion of Kasparov's bimodules $\mathcal{E}(\mathcal{A}_k, \mathcal{B}_k)$ which are projective and finite as right modules. This way, one obtains a first realization of the resulting category of non-commutative zero-dimensional motives in the abelian category of $k(\Lambda)$ -modules.

In general, given (X_α, S) , with $\{X_\alpha\}_{\alpha \in I}$ a projective system of Artin motives and S a semigroup of endomorphisms of $X = \varprojlim_{\alpha} X_\alpha$ as above, the datum of the semigroup action is encoded naturally by the *algebraic groupoid*

$$\mathcal{G} = X \rtimes S.$$

This is defined in the following way. One considers the Grothendieck group \tilde{S} of the abelian semigroup S . By using the injectivity of the partial action of S , one may also assume that S embeds in \tilde{S} . Then, the action of S on X extends to define a *partial action* of \tilde{S} . More precisely, for $s = \rho_2^{-1}\rho_1 \in \tilde{S}$ the two projections

$$E(s) = \rho_1^{-1}(\rho_2(1)\rho_1(1)), \quad F(s) = \rho_2^{-1}(\rho_2(1)\rho_1(1))$$

only depend on s and the map $s : A_{E(s)} \rightarrow A_{F(s)}$ defines an isomorphism of reduced algebras. It is immediate to verify that $E(s^{-1}) = F(s) = s(E(s))$ and that $F(ss') \geq F(s)s(F(s'))$. The algebraic groupoid \mathcal{G} is defined as the disjoint union

$$\mathcal{G} = \coprod_{s \in \tilde{S}} X^{F(s)}$$

which corresponds to the commutative direct-sum of reduced algebras

$$\bigoplus_{s \in \tilde{S}} A_{F(s)}.$$

The range and the source maps in \mathcal{G} are given *resp.* by the natural projection from \mathcal{G} to X and by its composition with the antipode \mathbf{S} which is defined, at the algebra level, by $\mathbf{S}(a)_s = s(a_{s^{-1}})$, $\forall s \in \tilde{S}$. The composition in the groupoid corresponds to the product of monomials $aU_s bU_t = as(b)U_{st}$.

Given two systems (X_α, S) and $(X'_{\alpha'}, S')$, with associated crossed-product algebras \mathcal{A}_k and \mathcal{B}_k and groupoids $\mathcal{G} = \mathcal{G}(X_\alpha, S)$ and $\mathcal{G}' = \mathcal{G}(X'_{\alpha'}, S')$ a *geometric correspondence* is given by a $(\mathcal{G}, \mathcal{G}')$ -space $Z = \text{Spec}(C)$, endowed with a right action of \mathcal{G}' which fulfills the following *étale* condition. Given a space such as \mathcal{G}' , that is a disjoint union of zero-dimensional pro-varieties over k , a right action of \mathcal{G}' on Z is given by a map $g : Z \rightarrow X'$ and a collection of partial isomorphisms

$$z \in g^{-1}(F(s)) \mapsto z \cdot s \in g^{-1}(E(s)) \quad (4.3)$$

fulfilling the following rules for partial action of the abelian group \tilde{S}

$$g(z \cdot s) = g(z) \cdot s, \quad z \cdot (ss') = (z \cdot s) \cdot s' \quad \text{on } g^{-1}(F(s) \cap s(F(s'))). \quad (4.4)$$

Here $x \mapsto x \cdot s$ denotes the partial action of \tilde{S} on X' . One checks that such an action gives to the k -linear space C a structure of right module over \mathcal{B}_k . The action of \mathcal{G}' on Z is étale if the corresponding module C is *finite and projective* over \mathcal{B}_k .

Given two systems (X_α, S) and $(X'_{\alpha'}, S')$ as above, an *étale correspondence* is therefore a $(\mathcal{G}(X_\alpha, S), \mathcal{G}(X'_{\alpha'}, S'))$ -space Z such that the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is étale.

The \mathbb{Q} -linear space of (*virtual*) *correspondences*

$$\text{Corr}((X_\alpha, S), (X'_{\alpha'}, S'))$$

is the rational vector space of formal linear combinations $U = \sum_i a_i Z_i$ of étale correspondences Z_i , modulo the relations arising from isomorphisms and equivalences: $Z \amalg Z' \sim Z + Z'$. The composition of correspondences is given by the fiber product over a groupoid. Namely, for three systems (X_α, S) , $(X'_{\alpha'}, S')$, $(X''_{\alpha''}, S'')$ joined by correspondences

$$(X_\alpha, S) \leftarrow Z \rightarrow (X'_{\alpha'}, S'), \quad (X'_{\alpha'}, S') \leftarrow W \rightarrow (X''_{\alpha''}, S''),$$

their composition is given by the rule

$$Z \circ W = Z \times_{\mathcal{G}'} W \tag{4.5}$$

that is the fiber product over the groupoid $\mathcal{G}' = \mathcal{G}(X'_{\alpha'}, S')$.

Finally, a system (X_α, S) as above is said to be *uniform* if the normalized counting measures μ_α on X_α satisfy $\xi_{\beta, \alpha} \mu_\alpha = \mu_\beta$.

Definition 4.1. *The category $\mathcal{E}\mathcal{V}^\circ(k)_K$ of algebraic endomotives with coefficients in a fixed extension K of \mathbb{Q} is the (pseudo)abelian category generated by the following objects and morphisms. The objects are uniform systems $M = (X_\alpha, S)$ of Artin motives over k , as above. The set of morphisms in the category connecting two objects $M = (X_\alpha, S)$ and $M' = (X'_{\alpha'}, S')$ is defined as*

$$\text{Hom}(M, M') = \text{Corr}((X_\alpha, S), (X'_{\alpha'}, S')) \otimes_{\mathbb{Q}} K.$$

The category $\mathcal{C}\mathcal{V}^\circ(k)_K$ of Artin motives embeds as a full sub-category in the category of algebraic endomotives

$$\iota : \mathcal{C}\mathcal{V}^\circ(k)_K \rightarrow \mathcal{E}\mathcal{V}^\circ(k)_K.$$

The functor ι maps an Artin motive $M = X$ to the system (X_α, S) with $X_\alpha = X$, $\forall \alpha$ and $S = \{\text{id}\}$.

4.1 Examples of algebraic endomotives

The category of algebraic endomotives is inclusive of a large and general class of examples of noncommutative spaces $\mathcal{A}_k = A \rtimes S$ which are described by semigroup actions on projective systems of Artin motives.

One may consider, for instance a pointed algebraic variety (Y, y_0) over a field k and a countable, unital, abelian semigroup S of *finite* endomorphisms of (Y, y_0) ,

unramified over $y_0 \in Y$. Then, there is a system (X_s, S) of Artin motives over k which is constructed from these data. More precisely, for $s \in S$, one sets

$$X_s = \{y \in Y : s(y) = y_0\}. \quad (4.6)$$

For a pair $s, s' \in S$, with $s' = sr$, the connecting map $\xi_{s,s'} : X_{sr} \rightarrow X_s$ is defined by

$$X_{sr} \ni y \mapsto r(y) \in X_s. \quad (4.7)$$

This is an example of a system indexed by the semigroup S itself, with partial order given by divisibility. One sets $X = \varprojlim_s X_s$.

Since $s(y_0) = y_0$, the base point y_0 defines a component Z_s of X_s for all $s \in S$. The pre-image $\xi_{s,s'}^{-1}(Z_s)$ in $X_{s'}$ is a union of components of $X_{s'}$. This defines a projection e_s onto an open and closed subset X^{e_s} of the projective limit X .

It is easy to see that the semigroup S acts on the projective limit X by partial isomorphisms $\beta_s : X \rightarrow X^{e_s}$ defined by the property

$$\beta_s : X \rightarrow X^{e_s}, \quad \xi_{su}(\beta_s(x)) = \xi_u(x), \quad \forall u \in S, \forall x \in X. \quad (4.8)$$

The map β_s is well-defined since the set $\{su : u \in S\}$ is cofinal and $\xi_u(x) \in X_{su}$, with $su\xi_u(x) = s(y_0) = y_0$. The image of β_s is in X^{e_s} , since by definition of β_s : $\xi_s(\beta_s(x)) = \xi_1(x) = y_0$. For $x \in X^{e_s}$, we have $\xi_{su}(x) \in X_u$. This shows that β_s defines an isomorphism of X with X^{e_s} , whose inverse map is given by

$$\xi_u(\beta_s^{-1}(x)) = \xi_{su}(x), \quad \forall x \in X^{e_s}, \forall u \in S. \quad (4.9)$$

The corresponding algebra morphisms ρ_s are then given by

$$\rho_s(f)(x) = f(\beta_s^{-1}(x)), \quad \forall x \in X^{e_s}, \quad \rho_s(f)(x) = 0, \quad \forall x \notin X^{e_s}. \quad (4.10)$$

This class of examples also fulfill an *equidistribution property*, making the uniform normalized counting measures μ_s on X_s compatible with the projective system and inducing a probability measure on the limit X . Namely, one has

$$\xi_{s',s}\mu_s = \mu_{s'}, \quad \forall s, s' \in S. \quad (4.11)$$

5 Analytic endomotives

In this section we assume that k is a number field. We fix an embedding $\sigma : k \hookrightarrow \mathbb{C}$ and we denote by \bar{k} an algebraic closure of $\sigma(k) \subset \mathbb{C}$ in \mathbb{C} .

When taking points over \bar{k} , algebraic endomotives yield 0-dimensional singular quotient spaces $X(\bar{k})/S$, which can be described by means of locally compact étale groupoids $\mathcal{G}(\bar{k})$ and the associated crossed product C^* -algebras $C(X(\bar{k})) \rtimes S$. This construction gives rise to the category of *analytic endomotives*.

One starts off by considering a uniform system (A_α, S) of Artin motives over k and the algebras

$$A_{\mathbb{C}} = A \otimes_k \mathbb{C} = \varinjlim_{\alpha} A_{\alpha} \otimes_k \mathbb{C}, \quad \mathcal{A}_{\mathbb{C}} = \mathcal{A}_k \otimes_k \mathbb{C} = A_{\mathbb{C}} \rtimes S. \quad (5.1)$$

The assignment

$$a \in A \rightarrow \hat{a}, \quad \hat{a}(\chi) = \chi(a) \quad \forall \chi \in X = \varprojlim_{\alpha} X_{\alpha} \quad (5.2)$$

defines an involutive embedding of algebras $A_{\mathbb{C}} \subset C(X)$. The C^* -completion $C(X)$ of $A_{\mathbb{C}}$ is an abelian AF C^* -algebra. One sets

$$\bar{\mathcal{A}}_{\mathbb{C}} = C(X) \rtimes S.$$

This is the C^* -completion of the algebraic crossed product $A_{\mathbb{C}} \rtimes S$. It is defined by the algebraic relations (4.1) with the involution which is apparent in the formulae (cf. [27],[28]).

In the applications that require to work with cyclic (co)homology, it is important to be able to restrict from C^* -algebras such as $\bar{\mathcal{A}}_{\mathbb{C}}$ to canonical dense subalgebras

$$\mathcal{A}_{\mathbb{C}} = C^{\infty}(X) \rtimes_{alg} S \subset \bar{\mathcal{A}}_{\mathbb{C}} \quad (5.3)$$

where $C^{\infty}(X) \subset C(X)$ is the algebra of locally constant functions. It is to this category of smooth algebras (rather than to that of C^* -algebras) that cyclic homology applies.

The following result plays an important role in the theory of endomotives and their applications to examples arising from the study of the thermodynamical properties of certain quantum statistical dynamical systems. We shall refer to the following proposition, in section 5.1 of this paper for the description of the properties of the ‘‘BC-system’’. The BC-system is a particularly relevant quantum statistical dynamical system which has been the prototype and the motivating example for the introduction of the notion of an endomotive. We refer to [7], § 4.1 for the definition of the notion and the properties of a state on a (unital) involutive algebra.

Proposition 5.1 ([7] Proposition 3.1). *1) The action (4.2) of G_k on $X(\bar{k})$ defines a canonical action of G_k by automorphisms of the C^* -algebra $\bar{\mathcal{A}}_{\mathbb{C}} = C(X) \rtimes S$, preserving globally $C(X)$ and such that, for any pure state φ of $C(X)$,*

$$\alpha \varphi(a) = \varphi(\alpha^{-1}(a)), \quad \forall a \in A, \quad \alpha \in G_k. \quad (5.4)$$

2) When the Artin motives A_{α} are abelian and normal, the subalgebras $A \subset C(X)$ and $\mathcal{A}_k \subset \bar{\mathcal{A}}_{\mathbb{C}}$ are globally invariant under the action of G_k and the states φ of $\bar{\mathcal{A}}_{\mathbb{C}}$ induced by pure states of $C(X)$ fulfill

$$\alpha \varphi(a) = \varphi(\theta(\alpha)(a)), \quad \forall a \in \mathcal{A}_k, \quad \theta(\alpha) = \alpha^{-1}, \quad \forall \alpha \in G_k^{ab} = G_k/[G_k, G_k] \quad (5.5)$$

On the totally disconnected compact space X , the abelian semigroup S of homeomorphisms acts, producing closed and open subsets $X^s \hookrightarrow X$, $x \mapsto x \cdot s$. The normalized counting measures μ_{α} on X_{α} define a *probability measure* on X with the property that the Radon–Nikodym derivatives

$$\frac{ds^* \mu}{d\mu} \quad (5.6)$$

are locally constant functions on X . One lets $\mathcal{G} = X \rtimes S$ be the corresponding étale locally compact groupoid. The crossed product C^* -algebra $C(X(\bar{k})) \rtimes S$ coincides with the C^* -algebra $C^*(\mathcal{G})$ of the groupoid \mathcal{G} .

The notion of right (or left) action of \mathcal{G} on a totally disconnected locally compact space \mathcal{Z} is defined as in the algebraic case by (4.3) and (4.4). A right action of \mathcal{G} on \mathcal{Z} gives on the space $C_c(\mathcal{Z})$ of continuous functions with compact support on \mathcal{Z} a structure of right module over $C_c(\mathcal{G})$. When the fibers of the map $g : \mathcal{Z} \rightarrow X$ are discrete (countable) subsets of \mathcal{Z} one can define on $C_c(\mathcal{Z})$ an inner product with values in $C_c(\mathcal{G})$ by

$$\langle \xi, \eta \rangle(x, s) = \sum_{z \in g^{-1}(x)} \bar{\xi}(z) \eta(z \circ s) \quad (5.7)$$

A right action of \mathcal{G} on \mathcal{Z} is *étale* if and only if the fibers of the map g are discrete and the identity is a compact operator in the right C^* -module $\mathcal{E}_{\mathcal{Z}}$ over $C^*(\mathcal{G})$ given by (5.7).

An *étale correspondence* is a $(\mathcal{G}(X_\alpha, S), \mathcal{G}(X'_{\alpha'}, S'))$ -space \mathcal{Z} such that the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is étale.

The \mathbb{Q} -vector space

$$\text{Corr}((X, S, \mu), (X', S', \mu'))$$

of linear combinations of étale correspondences \mathcal{Z} modulo the equivalence relation $\mathcal{Z} \cup \mathcal{Z}' = \mathcal{Z} + \mathcal{Z}'$ for disjoint unions, defines the space of (virtual) correspondences.

For $M = (X, S, \mu)$, $M' = (X', S', \mu')$, and $M'' = (X'', S'', \mu'')$, the composition of correspondences

$$\text{Corr}(M, M') \times \text{Corr}(M', M'') \rightarrow \text{Corr}(M, M''), \quad (Z, W) \mapsto Z \circ W$$

is given following the same rule as for the algebraic case (4.5), that is by the fiber product over the groupoid \mathcal{G}' . A correspondence gives rise to a bimodule $\mathcal{M}_{\mathcal{Z}}$ over the algebras $C(X) \rtimes S$ and $C(X') \rtimes S'$ and the composition of correspondences translates into the tensor product of bimodules.

Definition 5.2. *The category $C^*\mathcal{V}_K^0$ of analytic endomotives is the (pseudo)abelian category generated by objects of the form $M = (X, S, \mu)$ with the properties listed above and morphisms given as follows. For $M = (X, S, \mu)$ and $M' = (X', S', \mu')$ objects in the category, one sets*

$$\text{Hom}_{C^*\mathcal{V}_K^0}(M, M') = \text{Corr}(M, M') \otimes_{\mathbb{Q}} K. \quad (5.8)$$

The following result establishes a precise relation between the categories of Artin motives and (noncommutative) endomotives.

Theorem 5.3 ([7], Theorem 3.13). *The categories of Artin motives and algebraic and analytic endomotives are related as follows.*

- (1) *The map $\mathcal{G} \mapsto \mathcal{G}(\bar{k})$ determines a tensor functor*

$$\mathcal{F} : \mathcal{E}\mathcal{V}^0(k)_K \rightarrow C^*\mathcal{V}_K^0, \quad \mathcal{F}(X_\alpha, S) = (X(\bar{k}), S, \mu)$$

from algebraic to analytic endomotives.

- (2) The Galois group $G_k = \text{Gal}(\bar{k}/k)$ acts by natural transformations of \mathcal{F} .
- (3) The category $\mathcal{CV}^o(k)_K$ of Artin motives embeds as a full subcategory of $\mathcal{EV}^o(k)_K$.
- (4) The composite functor

$$c \circ \mathcal{F} : \mathcal{EV}^o(k)_K \rightarrow \mathcal{KK} \otimes K \quad (5.9)$$

maps the full subcategory $\mathcal{CV}^o(k)_K$ of Artin motives faithfully to the category $\mathcal{KK}_{G_k} \otimes K$ of G_k -equivariant KK -theory with coefficients in K .

Given two Artin motives $X = \text{Spec}(A)$ and $X' = \text{Spec}(B)$ and a component $Z = \text{Spec}(C)$ of the cartesian product $X \times X'$, the two projections turn C into a (A, B) -bimodule $c(Z)$. If $U = \sum a_i \chi_{Z_i} \in \text{Hom}_{\mathcal{CV}^o(k)_K}(X, X')$, $c(U) = \sum a_i c(Z_i)$ defines a sum of bimodules in $\mathcal{KK} \otimes K$. The composition of correspondences in $\mathcal{CV}^o(k)_K$ translates into the tensor product of bimodules in $\mathcal{KK}_{G_k} \otimes K$

$$c(U) \otimes_B c(L) \simeq c(U \circ L).$$

One composes the functor c with the natural functor $A \rightarrow A_{\mathbb{C}}$ which associates to a (A, B) -bimodule \mathcal{E} the $(A_{\mathbb{C}}, B_{\mathbb{C}})$ -bimodule $\mathcal{E}_{\mathbb{C}}$. The resulting functor

$$c \circ \mathcal{F} \circ \iota : \mathcal{CV}^o(k)_K \rightarrow \mathcal{KK}_{G_k} \otimes K$$

is faithful since a correspondence such as U is uniquely determined by the corresponding map of K -theory $K_0(A_{\mathbb{C}}) \otimes K \rightarrow K_0(B_{\mathbb{C}}) \otimes K$.

5.1 The endomotive of the BC system

The prototype example of the data which define an analytic endomotive is the system introduced by Bost and Connes in [6]. The evolution of this C^* -dynamical system encodes in its group of symmetries the arithmetic of the maximal abelian extension of $k = \mathbb{Q}$.

This quantum statistical dynamical system is described by the datum given by a noncommutative C^* -algebra of *observables* $\bar{A}_{\mathbb{C}} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes_{\alpha} \mathbb{N}^{\times}$ and by the *time evolution* which is assigned in terms of a one-parameter family of automorphisms σ_t of the algebra. The action of the (multiplicative) semigroup $S = \mathbb{N}^{\times}$ on the commutative algebra $C^*(\mathbb{Q}/\mathbb{Z}) \simeq C(\hat{\mathbb{Z}})$ is defined by

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho), & \text{if } \rho \in n\hat{\mathbb{Z}} \\ 0, & \text{otherwise} \end{cases}, \quad \rho \in \hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}.$$

For the definition of the associate endomotive, one considers the projective system $\{X_n\}_{n \in \mathbb{N}}$ of zero-dimensional algebraic varieties $X_n = \text{Spec}(A_n)$, where $A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$ is the group ring of the abelian group $\mathbb{Z}/n\mathbb{Z}$. The inductive limit $A = \varinjlim_n A_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ is the group ring of \mathbb{Q}/\mathbb{Z} . The endomorphism $\rho_n : A \rightarrow A$ associated to an element $n \in S$ is given on the canonical basis $e_r \in \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$,

$r \in \mathbb{Q}/\mathbb{Z}$, by

$$\rho_n(e_r) = \frac{1}{n} \sum_{ns=r} e_s. \quad (5.10)$$

The Artin motives X_n are normal and abelian, so that Proposition 5.1 applies. The action of the Galois group $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $X_n = \text{Spec}(A_n)$ is obtained by composing a character $\chi : A_n \rightarrow \overline{\mathbb{Q}}$ with the action of an element $g \in G_{\mathbb{Q}}$. Since χ is determined by the n -th root of unity $\chi(e_{1/n})$, this implies that the action of $G_{\mathbb{Q}}$ factorizes through the cyclotomic action and coincides with the *symmetry group* of the BC-system. The subalgebra $\mathcal{A}_{\mathbb{Q}} \subset \overline{\mathcal{A}}_{\mathbb{C}} = C(X) \rtimes S$ coincides with the rational subalgebra defined in [6].

There is an interesting description of this system in terms of a pointed algebraic variety (Y, y_0) (cf. section 4.1) on which the abelian semigroup S acts by finite endomorphisms. One considers the pointed affine group scheme $(\mathbb{G}_m, 1)$ (the multiplicative group) and lets S be the semigroup of non-zero endomorphisms of \mathbb{G}_m . These endomorphisms correspond to maps of the form $u \mapsto u^n$, for some $n \in \mathbb{N}$. Then, the general construction outlined in section 4.1 determines on $(\mathbb{G}_m(\mathbb{Q}), 1)$ the BC system.

One considers the semigroup $S = \mathbb{N}^{\times}$ acting on $\mathbb{G}_m(\mathbb{Q})$ as specified above. It follows from the definition (4.6) that $X_n = \text{Spec}(A_n)$ where

$$A_n = \mathbb{Q}[u_n^{\pm 1}]/(u_n^n - 1).$$

For $n|m$ the connecting morphism $\xi_{m,n} : X_m \rightarrow X_n$ is defined by the algebra homomorphism $A_n \rightarrow A_m$, $u_n^{\pm 1} \mapsto u_m^{\pm a}$ with $a = m/n$. Thus, one obtains an isomorphism of \mathbb{Q} -algebras

$$\iota : A = \varinjlim_n A_n \xrightarrow{\sim} \mathbb{Q}[\mathbb{Q}/\mathbb{Z}], \quad \iota(u_n) = e_{\frac{1}{n}}. \quad (5.11)$$

The partial isomorphisms $\rho_n : \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rightarrow \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ of the group ring as described by the formula (5.10) correspond under the isomorphism ι , to those given by (4.8) on $X = \varinjlim_n X_n$. One identifies X with its space of characters $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rightarrow \overline{\mathbb{Q}}$. Then, the projection $\xi_m(x)$ is given by the restriction of (the character associated to) $x \in X$ to the subalgebra A_m . The projection of the composite of the endomorphism ρ_n of (5.10) with $x \in X$ is given by

$$x(\rho_n(e_r)) = \frac{1}{n} \sum_{ns=r} x(e_s).$$

This projection is non-zero if and only if the restriction $x|_{A_n}$ is the trivial character, that is if and only if $\xi_n(x) = 1$. Moreover, in that case one has

$$x(\rho_n(e_r)) = x(e_s), \quad \forall s, \quad ns = r,$$

and in particular

$$x(\rho_n(e_{\frac{1}{k}})) = x(e_{\frac{1}{nk}}). \quad (5.12)$$

For $k|m$ the inclusion of algebraic spaces $X_k \subset X_m$ is given at the algebra level by the surjective homomorphism

$$j_{k,m} : A_m \rightarrow A_k, \quad j_{k,m}(u_m) = u_k.$$

Thus, one can rewrite (5.12) as

$$x \circ \rho_n \circ j_{k,nk} = x|_{A_{nk}}. \quad (5.13)$$

This means that

$$\xi_{nk}(x) = \xi_k(x \circ \rho_n).$$

By using the formula (4.9), one obtains the desired equality of the ρ 's of (5.10) and (4.10).

This construction continues to make sense for the affine algebraic variety $\mathbb{G}_m(k)$ for any field k , including the case of a field of positive characteristic. In this case one obtains new systems, different from the BC system.

6 Applications: the geometry of the space of adèles classes

The functor

$$\mathcal{F} : \mathcal{E}\mathcal{V}^o(k)_K \rightarrow C^*\mathcal{V}_K^o$$

which connects the categories of algebraic and analytic endomotives establishes a significant bridge between the commutative world of Artin motives and that of non-commutative geometry. When one moves from commutative to noncommutative algebras, important new tools of thermodynamical nature become available. One of the most relevant techniques (for number-theoretical applications) is supplied by the theory of Tomita and Takesaki for von Neumann algebras ([36]) which associates to a suitable state φ (*i.e.* a faithful weight) on a von Neumann algebra M , a one-parameter group of automorphisms of M (*i.e.* the modular automorphisms group)

$$\sigma_t^\varphi : \mathbb{R} \rightarrow \text{Aut}(M), \quad \sigma_t^\varphi(x) = \Delta_\varphi^{it} x \Delta_\varphi^{-it}.$$

Δ_φ is the modular operator which acts on the completion $L^2(M, \varphi)$ of $\{x \in M : \varphi(x^*x) < \infty\}$, for the scalar product $\langle x, y \rangle = \varphi(y^*x)$.

This general theory applies in particular to the unital involutive algebras $\mathcal{A} = C^\infty(X) \rtimes_{alg} S$ of (5.3) and to the related C^* -algebras $\bar{\mathcal{A}}_{\mathbb{C}}$ which are naturally associated to an endomotive.

A remarkable result proved by Connes in the theory and classifications of factors ([8]) states that, modulo inner automorphisms of M , the one-parameter family σ_t^φ is *independent* of the choice of the state φ . This way, one obtains a *canonically defined* one parameter group of automorphisms classes

$$\delta : \mathbb{R} \rightarrow \text{Out}(M) = \text{Aut}(M)/\text{Inn}(M).$$

In turn, this result implies that the crossed product dual algebra

$$\hat{M} = M \rtimes_{\sigma_t^\varphi} \mathbb{R}$$

and the dual scaling action

$$\theta_\lambda : \mathbb{R}_+^* \rightarrow \text{Aut}(\hat{M}) \tag{6.1}$$

are *independent* of the choice of (the weight) φ .

When these results are applied to the analytic endomotive $\mathcal{F}(X_\alpha, S)$ associated to an algebraic endomotive $M = (X_\alpha, S)$, the above dual representation of \mathbb{R}_+^* combines with the representation of the absolute Galois group G_k . In the particular case of the endomotive associated to the BC-system (*cf.* section 5.1), the resulting representation of $G_{\mathbb{Q}} \times \mathbb{R}_+^*$ on the cyclic homology HC_0 of a suitable $\mathbb{Q}(\Lambda)$ -module $D(\mathcal{A}, \varphi)$ associated to the thermodynamical dynamics of the system $(\mathcal{A}, \sigma_t^\varphi)$ determines the spectral realization of the zeroes of the Riemann zeta-function and of the Artin L -functions for abelian characters of G_k (*cf.* [7], Theorem 4.16).

The action of the group $W = G_{\mathbb{Q}} \times \mathbb{R}_+^*$ on the cyclic homology $HC_0(D(\mathcal{A}, \varphi))$ of the noncommutative motive $D(\mathcal{A}, \varphi)$ is analogous to the action of the Weil group on the étale cohomology of an algebraic variety. In particular, the action of \mathbb{R}_+^* is the ‘characteristic zero’ analog of the action of the (geometric) Frobenius on étale cohomology. This construction determines a functor

$$\omega : \mathcal{E}\mathcal{V}^o(k)_K \rightarrow \text{Rep}_{\mathbb{C}}(W)$$

from the category of endomotives to the category of (infinite-dimensional) representations of the group W .

The analogy with the Tannakian formalism of classical motive theory is striking. It is also important to underline the fact that the whole thermodynamical construction is non-trivial and relevant for number-theoretic applications only because of the particular nature of the factor M (type III₁) associated to the original datum (\mathcal{A}, φ) of the BC-system.

It is tempting to compare the original choice of the state φ (weight) on the algebra \mathcal{A} which singles out (via the Gelfand-Naimark-Segal construction) the factor M defined as the weak closure of the action of $\bar{\mathcal{A}} = C(X) \rtimes S$ in the Hilbert space $\mathcal{H}_\varphi = L^2(M, \varphi)$, with the assignment of a factor $(X, p, m)^r$ ($r \in \mathbb{Z}$) on a pure motive $M = (X, p, m)$: *cf.* (2.8). In classical motive theory, one knows that the assignment of a \mathbb{Z} -grading is canonical only for homological equivalence or under the assumption of the Standard Conjecture of Künneth type. In fact, the definition of a weight structure depends upon the definition of a complete system of orthogonal central idempotents π_X^i .

Passing from the factor M to the canonical dual representation (6.1) carries also the advantage to work in a setting where projectors are classified by their real dimension (\hat{M} is of type II₁), namely in a noncommutative framework of continuous geometry which generalizes and yet still retains some relevant properties of the algebraic correspondences (*i.e.* degree or dimension).

The process of dualization is in fact subsequent to a thermodynamical “cooling procedure” in order to work with a system whose algebra approaches and becomes in the limit, a commutative algebra (*i.e.* I_∞). Finally, one has also to implement a

further step in which one filters (*i.e.* “distils”) the relevant noncommutative motive $D(\mathcal{A}, \varphi)$ within the derived framework of cyclic modules (*cf.* section 3.1). This procedure is somewhat reminiscent of the construction of the vanishing cohomology in algebraic geometry (*cf.* [23]).

When the algebra of the BC-system gets replaced by the noncommutative algebra of coordinates $\mathcal{A} = \mathcal{S}(\mathbb{A}_k) \rtimes k^*$ of the adèle class space $X_k = \mathbb{A}_k/k^*$ of a number field k , the cooling procedure is described by a restriction morphism of (rapidly decaying) functions on X_k to functions on the “cooled down” subspace C_k of idèle classes (*cf.* [7], Section 5). In this context, the representation of C_k on the cyclic homology $HC_0(\mathcal{H}_{k,\mathbb{C}}^1)$ of a suitable noncommutative motive $\mathcal{H}_{k,\mathbb{C}}^1$ produces the spectral realization of the zeroes of Hecke L -functions (*cf. op.cit.*, Theorem 5.6).

The whole construction describes also a natural way to associate to a noncommutative space a canonical set of “classical points” which represents the analogue in characteristic zero, of the geometric points $C(\overline{\mathbb{F}}_q)$ of a smooth, projective curve C/\mathbb{F}_q .

7 Bibliography

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