ENTIRE DOWNWARD TRANSLATING SOLITONS TO THE MEAN CURVATURE FLOW IN MINKOWSKI SPACE

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Abstract. In this paper, we study entire translating solutions $u(x)$ to a mean curvature flow equation in Minkowski space. We show that if $\Sigma = \{(x, u(x))|x \in \mathbb{R}^n\}$ is a strictly spacelike hypersurface, then Σ reduces to a strictly convex rank k soliton in $\mathbb{R}^{k,1}$ (after splitting off trivial factors) whose "blowdown" converges to a multiple $\lambda \in (0, 1)$ of a positively homogeneous degree one convex function in $\mathbb{R}^k$. We also show that there is nonuniqueness as the rotationally symmetric solution may be perturbed to a solution by an arbitrary smooth order one perturbation.

1. Introduction

Let $\mathbb{R}^{n,1}$ be the Minkowski space with Lorentz metric

$$\bar{g} = \sum_{i=1}^{n} dx_i^2 - dx_{n+1}^2.$$

We will say that a hypersurface $\Sigma = \{(x, u(x))|x \in \Omega\} \subset \mathbb{R}^{n,1}$ is strictly spacelike if $u \in C^1(\Omega)$ and $|Du| \le c_0 < 1$ in Ω.

Ecker and Huisken [5] studied the mean curvature flow with forcing term in cosmological spacetimes V and constructed a spacelike hypersurface with prescribed mean curvature in V. More specifically, they studied the parabolic evolution equation

$$\frac{\partial}{\partial t}\mathbf{F} = (H - \mathcal{H})\nu, \tag{1.1}$$

where $\mathbf{F}(p, 0) : M^n \to V$ is a compact n-dimensional manifold and $\mathcal{H}$ is the forcing term. Later, M. Aarons [1] proved the following convergence result.

Theorem 1.1. *([1]) Let M_0 be a smooth spacelike hypersurface with bounded curvature. Suppose M_0 never intersects future null infinity I^+ or past null infinity I^-. Then M_t converges under the flow*

$$\frac{\partial u}{\partial t} = \sqrt{1-|Du|^2}\left[div\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) - c\right], \tag{1.2}$$

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to a convex downward translating soliton, that is, an entire solution of

$$H = c + a\frac{1}{\sqrt{1-|\nabla u|^2}}, \tag{1.3}$$

where $-c \le a < 0$.

It's easy to see that when $a = -c$, the only convex translating soliton is the trivial one. Therefore, in the following, we will focus on the case where $-c < a < 0$.

After rescaling, we may assume $a = -1$ and consider $H = C - \frac{1}{\sqrt{1-|Du|^2}}$, where $C > 1$ is a constant. We obtain

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|Du|^2}}\right) = C - \frac{1}{\sqrt{1-|Du|^2}} \tag{1.4}$$

or in nondivergence form

$$\left(\delta_{ij} + \frac{u_i u_j}{1-|Du|^2}\right) u_{ij} = C\sqrt{1-|Du|^2} - 1. \tag{1.5}$$

Notice that any $u = \vec{v}\cdot x + b$, $|v| = \sqrt{1-\frac{1}{C^2}}$ (a maximal hypersurface) is a solution of (1.5). The existence of a unique (up to translation) radially symmetric solution of (1.5) was shown by Ju, Lu and Jian [9].

Aarons [1] in fact conjectured that any solution u of (1.3) is either rotationally symmetric about some point x_0 or is a hyperplane. However, this conjecture is not correct when $-c < a < 0$. Let $x' = (x_1, \ldots, x_k)$ and set $u(x) = \sum_{i=k+1}^n a_i x_i + h(x')$ where h is strictly convex in x' and $a = (a_{k+1}, \ldots, a_n)$ is chosen such that $C\sqrt{1-|a|^2} > 1$. Then u satisfies (1.5) if and only if h satisfies

$$\sum_{i,j=1}^k (\delta_{ij} + \frac{h_i h_j}{1-|a|^2-|Dh|^2})h_{ij} = C\sqrt{1-|a|^2}\sqrt{1-\frac{|Dh|^2}{1-|a|^2}} - 1\ . \tag{1.6}$$

Now let $\tilde h = \frac{1}{\lambda^2}h(\lambda x)$, $\tilde C = \lambda C > 1$ with $\lambda = \sqrt{1-|a|^2}$. Then $\tilde h$ satisfies

$$\sum_{i,j=1}^k (\delta_{ij} + \frac{\tilde h_i \tilde h_j}{1-|D\tilde h|^2})\tilde h_{ij} = \tilde C\sqrt{1-|D\tilde h|^2} - 1\ . \tag{1.7}$$

Thus $\tilde h$ is a rank k solution of (1.5) in $\mathbb{R}^k$ with C replaced by $\tilde C = \lambda C > 1$.

In fact, we will show a splitting theorem analogous to what Choi-Treibergs [4] proved for spacelike constant mean curvature hypersurfaces.

Theorem 1.2. *Let u be a strictly spacelike solution of* (1.4) *and let* $\Sigma = \{(x, u(x))|x \in \mathbb{R}^n\}$ *be the graph of u. Then Σ is convex with uniformly bounded second fundamental form. Moreover after an $R^{n,1}$ rigid motion, $R^{n,1}$ splits as a product $R^{k,1} \times R^{n-k}$ such that Σ also splits as a product $\Sigma^k \times R^{n-k}$ where $\Sigma^k \subset R^{k,1}$ is a strictly convex graphical solution in $\mathbb{R}^{k,1}$*

Thus it is natural to ask if Aarons conjecture is correct for u, a strictly convex solution of (1.4) in $\mathbb{R}^n$? In other words, is $\Sigma = \{(x, u(x))|x \in \mathbb{R}^n\}$ rotationally symmetric? The answer is no.

Theorem 1.3. *Let* $f \in C^2(S^{n-1}_{\tilde{C}})$, $\tilde{C} = \sqrt{1 - (\frac{1}{C})^2}$. *Then there exists an entire strictly spacelike hypersurface u satisfying equation* (1.4) *such that*

$$u(x) \to \tilde{C}|x| - \frac{n-1}{C^2} \log|x| + f(\tilde{C}x) \text{ as } |x| \to \infty.$$

As in the work of Treibergs [12] and Choi-Treibergs [4], the blow-down of a convex strictly spacelike solution $V_u = \lim_{r\to\infty} \frac{u(rx)}{r}$ converges uniformly on compact subsets to the space $\tilde{C}\mathcal{Q}$ of convex homogeneous degree one convex functions whose gradient has magnitude $\tilde{C}$ wherever defined. It was shown in [4] that the space $\mathcal{Q}$ is in one to one correspondence with the set of *lightlike* directions

$$L_u := \{x \in S^{n-1} : V_u(x) = 1\} .$$

It may be possible that any cone in $\tilde{C}\mathcal{Q}$ arises as the blow-down of a solution to (1.4) but we have not shown this.

An outline of the paper is as follows. In section 2 we show the strictly space like assumption implies that the graph Σ is mean convex. Then in section 3 we show Σ is in fact convex and then prove the splitting Theorem 1.2. In section 4 we study the blow-down V_u and finally in section 5 following [12], we construct counterexamples for the radial cone in $\tilde{C}\mathcal{Q}$ and prove Theorem 1.3

2. Strictly spacelike implies mean convex

Let $a^{ij} = \delta_{ij} + \frac{u_i u_j}{w^2}$, where $w = (1 - |Du|^2)^{1/2}$; then equation (1.5) becomes

$$a^{ij}u_{ij} = Cw - 1 \tag{2.1}$$

Then $w_i = -\frac{u_k u_{ki}}{w}$ and

$$
\begin{aligned}
w_{ij} &= -\frac{u_{ki}u_{kj}}{w} - \frac{u_k u_{kij}}{w} + \frac{u_k u_{ki} w_j}{w^2} \\
&= -\frac{u_k u_{kij}}{w} - \frac{1}{w}\left(u_{ki}u_{kj} + \frac{u_k u_{ki} u_l u_{lj}}{w^2}\right) \\
&= -\frac{u_k u_{kij}}{w} - \frac{1}{w} a^{kl} u_{ki} u_{lj}.
\end{aligned} \tag{2.2}
$$

Lemma 2.1. *Suppose $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$ is a strictly spacelike hypersurface, and $u(x)$ satisfies equation* (1.5)*. Then Σ is mean convex, that is $H \geq 0$.*

Proof. We differentiate equation (2.1) with respect to x_k to obtain

$$
(a^{ij})_k u_{ij} + a^{ij} u_{ijk} = C w_k. \tag{2.3}
$$

Since

$$
\begin{aligned}
(a^{ij})_k u_{ij} &= \left(\frac{u_{ik}u_j}{w^2} + \frac{u_i u_{jk}}{w^2} - 2\frac{u_i u_j w_k}{w^3}\right) u_{ij} \\
&= \frac{2}{w}\left(\frac{u_{ik}u_j}{w} + \frac{u_i u_j u_l u_{lk}}{w^3}\right) u_{ij} \\
&= \frac{2}{w}\left(-w_i u_{ik} - \frac{u_i u_l u_{lk}}{w^2} w_i\right) \\
&= -\frac{2}{w}\left(\delta_{ij} + \frac{u_i u_j}{w^2}\right) w_i u_{kj} \\
&= -\frac{2}{w} a^{ij} w_i u_{kj},
\end{aligned} \tag{2.4}
$$

and $u_{ijk} = u_{kij}$, this gives

$$
a^{ij} u_{kij} - \frac{2}{w} a^{ij} w_i u_{kj} = C w_k. \tag{2.5}
$$

Multiplying (2.5) by $\frac{u_k}{w}$ and using $\frac{u_k u_{kij}}{w} = -w_{ij} - \frac{1}{w} a^{kl} u_{ki} u_{lj}$, we obtain

$$
a^{ij} w_{ij} - 2\frac{a^{ij} w_i w_j}{w} + C\frac{u_k}{w} w_k = -\frac{1}{w} a^{ij} a^{kl} u_{ki} u_{lj}. \tag{2.6}
$$

We now observe that since $|A|^2 = \frac{1}{w^2} a^{ij} a^{kl} u_{ki} u_{lj}$ we can rewrite (2.6) as

$$
a^{ij}(\frac{1}{w})_{ij} + C\frac{u_k}{w}(\frac{1}{w})_k = |A|^2 \frac{1}{w} \ . \tag{2.7}
$$

The Omori-Yau maximum principle (see for example [13], [11]) implies that $\frac{1}{w}$ achieves its maximum at infinity and moreover, there exists a sequence $\{P_N\}$ such that $\frac{1}{w}(P_N) \to \sup \frac{1}{w}$, $|\nabla(\frac{1}{w})|(P_N) < 1/N$, and $(\frac{1}{w})_{ij}(P_N) \geq -1/N\delta_{ij}$. Therefore,

$$
\frac{1}{n} H^2 \frac{1}{w} \leq |A|^2 \frac{1}{w} \leq \frac{C_1}{N}\frac{1}{w} \quad \text{at } P_N. \tag{2.8}
$$

Thus $H(P_N) \to 0$ at infinity. Since $H = C - \frac{1}{w}$ we obtain $\inf H = 0$. $\square$

3. Mean convexity implies convexity and constant rank

In this section, we will use ideas due to Hamilton [7] to prove that under the strictly spacelike assumption, Σ is in fact convex. We use the following approximation of Heidusch[8].

Definition 3.1. The δ-approximation to the function $\min(x_1, x_2)$ is given by

$$\mu_2(x_1, x_2) = \frac{x_1 + x_2}{2} - \sqrt{\left(\frac{x_1 - x_2}{2}\right)^2 + \delta^2}$$

for any $\delta > 0$. The δ-approximation to the function $\min(x_1, x_2, \cdots, x_n)$ is defined recursively by

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} \mu_2(x_i,\ \mu_{n-1}(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n))$$

The following lemma is elementary (see [1]).

Lemma 3.2. *For every $\delta > 0$ and $n \geq 2$ we have:*
1. μ_n is smooth, symmetric, monotonically increasing and concave.
2. $\frac{\partial \mu_n}{\partial x_i} \leq 1$.
3. $\min(x_1, \cdots, x_n) - n\delta \leq \mu_n \leq \min(x_1, \cdots, x_n)$.
4. For $x \in \mathbb{R}^n$ we have

$$\mu_n \leq \sum_{i=1}^{n} \frac{\partial \mu_n}{\partial x_i} x_i \leq \mu_n + n\delta,$$

and $\sum_{i=1}^{n} \frac{\partial \mu_n}{\partial x_i} x_i^2 \geq \mu_n^2 - n\delta^2 - \frac{n\delta}{4} \sum_{1 \leq i < j \leq n} |x_i + x_j|$.

Lemma 3.3. *Assume $\Sigma = \{(x, u(x)) | x \in \mathbb{R}^n\}$ is mean convex, and u satisfies equation (1.5). Then the principal curvatures of A are nonnegative, i.e. Σ is convex.*

Proof. Let p be a fixed point in Σ (we may assume $p = (0, 0)$) and let r be the distance function from p restricted to the geodesic ball $B^{\Sigma}(p, a)$ of radius a centered at p (in the induced metric on Σ). Let $f(x) = |A|^2 = \sum_{i,j} h_{ij}^2$. By a well-known calculation (see equation (2.24) of [3])

$$\frac{1}{2}\triangle\left(\sum_{i,j} h_{ij}^2 =\right) = \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j} h_{ij} H_{ij} + \left(\sum_{i,j} h_{ij}^2\right)^2 - \left(\sum_{i,j,k} h_{ij} h_{jm} h_{mi}\right) H \tag{3.1}$$

$$
\begin{aligned}
H_{ij} &= \nabla_i \nabla_j H = \nabla_i \left(\nabla_j \langle \nu, e_{n+1}\rangle\right) \\
&= \nabla_i \langle h_{jk}\tau_k, e_{n+1}\rangle = -h_{ijk}u_k - h_{ik}h_{jk}\nu^{n+1}.
\end{aligned} \tag{3.2}
$$

In the following, we will denote ν^{n+1} by V.

$$
\begin{aligned}
\frac{1}{2}\triangle f &= \sum_{i,j,k} h_{ijk}^2 - h_{ij}h_{ijk}u_k - h_{ij}h_{ik}h_{jk}V + f^2 - \left(\sum_{i,j,m} h_{ij}h_{jm}h_{mi}\right)(C - V) \\
&\geq -\frac{f(V^2-1)}{4} + f^2 - Cf^{3/2} \\
&\geq -\frac{C^2}{4}f + f^2 - Cf^{3/2} \geq \frac{1}{2}f^2 - C_1
\end{aligned} \tag{3.3}
$$

Let $\eta(x) = a^2 - r^2$ and set $g = \eta^2 f$. Then in $B^{\Sigma}(p, a)$,

$$
\frac{1}{2}(\eta^{-2}g)^2 \leq C_1 + \Delta(\eta^{-2}g) = C_1 + \eta^{-2}\Delta g - 2\eta^{-3} < \nabla\eta, \nabla g > + g\Delta(\eta^{-2})\ . \tag{3.4}
$$

At the point $\overline{x}$ where g assumes its maximum, $\nabla g = 0$ and $\Delta g \leq 0$. Since $R_{ii} \geq -\frac{H^2}{4} \geq -\frac{C^2}{4}$, we have by Lemma 1 of [13] that $\Delta r^2 \leq C_3(1 + r^2)$. Hence at $\overline{x}$,

$$
\begin{aligned}
\frac{1}{2}g^2 &\leq C_1\eta^4 + g\eta^4\Delta(\eta^{-2}) = C_1\eta^4 - 2g\eta\Delta\eta + 6g|\nabla\eta|^2 \\
&\leq C_2(a^8 + 2g((a^2)\Delta r^2 + 12r^2)) \leq C_4(a^8 + a^4 g)
\end{aligned} \tag{3.5}
$$

It follows that $g(\overline{x}) \leq C_5 a^4$. Therefore we can let $a \to \infty$ to conclude $|A|^2 \leq C_5$.

Next we will show that the smallest principal curvature $\lambda_{\min}$ of Σ is nonnegative. Let $\mu_n(\lambda_1, \cdots, \lambda_n) = F(\gamma^{ik}h_{kl}\gamma^{lj})$, assume μ_n achieves its minimum at an interior point x_0. Then at this point we have

$$
\begin{aligned}
\triangle\mu_n &= F^{ij}h_{ijkk} + F^{rl,st}h_{rlk}h_{stk} \\
&= F^{ij}(H_{ij} - Hh_{ij}^2 + h_{ij}h_{kk}^2) + F^{rl,st}h_{rlk}h_{stk} \\
&\leq F^{ij}\nabla_k h_i^j \langle \tau_k, e_{n+1}\rangle - F^{ij}h_i^k h_{jk}\nu^{n+1} - HF^{ij}h_{ij}^2 + (\mu_n + n\delta)|A|^2 \\
&\leq \langle \nabla_k \mu_n, e_{n+1}\rangle - \mu_n^2 + n\delta^2 + \frac{n\delta}{4}\sum_{1\leq i<j\leq n}|\lambda_i + \lambda_j| \\
&+ H\left(-\mu_n^2 + n\delta^2 + \frac{n\delta}{4}\sum_{1\leq i<j\leq n}|\lambda_i + \lambda_j|\right) + (\mu_n + n\delta)|A|^2.
\end{aligned} \tag{3.6}
$$

Thus,

$$
0 \leq -\mu_n^2 + (\mu_n + n\delta)|A|^2 + n\delta^2 + \frac{n\delta}{4}\sum_{1\leq i<j\leq n}|\lambda_i + \lambda_j|. \tag{3.7}
$$

Letting $\delta \to 0$ we find,

$$\lambda_{\min}^2 \leq \lambda_{\min}|A|^2, \tag{3.8}$$

which implies that $\lambda_{\min} \geq 0$.

Since we have already proven that $|A|^2$ is bounded, we can again apply the Omori-Yau maximum principle (this time on Σ) and show that, if μ_n achieves its minimum at infinity then $\mu_n \geq 0$. This completes the proof that mean convexity implies convexity. $\square$

Now that we have proved the convexity of Σ, we prove the splitting Theorem 1.2 of the introduction.

Proof of Theorem 1.2. Suppose that for some unit vector $\vec{v}$ and some $x_0 \in \mathbb{R}^n$, $D^2_{\vec{v}}u(x_0) = 0$. Applying an isometry (boost transformation) of $\mathbb{R}^{n,1}$, may assume $x_0 = 0$, $\vec{v} = e_n$, $Du(0) = 0$, $u_{nn}(0) = 0$ and u_{ij} is nonnegative. Rewrite (1.5) as

$$\Delta u = -\frac{u_i u_j}{1-|Du|^2} u_{ij} + C\sqrt{1-|Du|^2} - 1 \tag{3.9}$$

Differentiating (3.9) twice in the x_n direction, we can apply the argument of Korevaar (a special case of [10]) exactly as in Theorem 3.1 of Choi-Treibergs [4] to conclude $u_{nn} \equiv 0$ and Σ is ruled by lines parallel to the x_n axis. Therefore $\Sigma = \Sigma^{n-1} \times \mathbb{R}^1$ and also $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$. Since u was arranged to be nonnegative, $u = h(x')$, $x' = (x_1, \ldots, x_{n-1})$ where

$$\sum_{i,j=1}^{n-1} (\delta_{ij} + \frac{h_i h_j}{1-|Dh|^2}) h_{ij} = C\sqrt{1-|Dh|^2} \, - 1 \ . \tag{3.10}$$

Proceeding inductively completes the proof of Theorem 1.2.

4. The asymptotic cone at infinity.

In this section, we will study the asymptotic behavior of u at infinity.

Proposition 4.1. *Let u be a convex space like solution of* (1.5)*. Assume $u(0) = 0$ and denote $u^h(x) = \frac{u(hx)}{h}$. Define $V_u(x) = \lim_{h\to\infty} u^h(x)$ then $V_u(x)$ exists for all x and is a positively homogeneous degree one convex function. Moreover for all $x \in \mathbb{R}^n$ and $\delta > 0$ there exists $y \in \mathbb{R}^n$ so that $|y-x| = \delta$ and $|V_u(x) - V_u(y)| = \sqrt{1-\frac{1}{C^2}}\,\delta$. In particular $|DV_u(x)| = \sqrt{1-\frac{1}{C^2}}$ at every point of differentiability of $V_u(x)$.*

Proof. Note that since u is convex, $0 = u(0) \geq u(hx) - \sum_{i=1}^{n} hx_i u_{x_i}(hx)$ we get $\frac{d}{dh}u^h(x) \geq 0$. Then $V_u(x)$, the projective boundary values (blow-down) of u at infinity in the terminology of Treibergs [12] and Choi-Treibergs [4], is well-defined, strictly spacelike, convex on $\mathbb{R}^n$ and satisfies

$$V_u(\lambda x) = \lambda V_u(x), \quad \lambda > 0,$$

$$|V_u(x) - V_u(y)| \leq \sqrt{1 - \frac{1}{C^2}}\ |x - y|\ .$$

Claim: for all $x \in \mathbb{R}^n$ and $\delta > 0$ there exists $y \in \mathbb{R}^n$ so that $|y - x| = \delta$ and $|V_u(x) - V_u(y)| = \sqrt{1 - \frac{1}{C^2}}\ \delta$. Suppose the claim is false. Then there exists $x \in \mathbb{R}^n$ and $\varepsilon > 0$ such that

$$V_u(y) \leq V_u(x) + (1 - 2\varepsilon)\sqrt{1 - \frac{1}{C^2}}\ \delta \ \ \forall y \in \partial B(x, \delta)\ .$$

Since $u^h(x) \to V_u(x)$ uniformly on compact subsets, we may choose h_0 large so that for all $h > h_0$,

$$u^h(y) \leq V_u(x) + (1 - \varepsilon)\sqrt{1 - \frac{1}{C^2}}\ \delta \ \ \forall y \in \partial B(x, \delta)\ .$$

Now $u^h(y)$ satisfies

$$H^h := \operatorname{div}\left(\frac{Du^h}{\sqrt{1 - |Du^h|^2}}\right) = h(C - \frac{1}{\sqrt{1 - |Du^h|^2}}) \geq 0 \ \text{ in } B(x, \delta) \tag{4.1}$$

We now make use of the radial solutions of the maximal surface equation $H = 0$ introduced by Bartnik and Simon [2]. Consider the barrier

$$w(y) = V_u(x) + (1-\varepsilon)\sqrt{1 - \frac{1}{C^2}}\ \delta + \int_0^{|y-x|} \frac{h}{\sqrt{t^{2n-2} + h^2}}\, dt - \sqrt{1 - \frac{1}{C^2}} \int_0^{\delta} \frac{h}{\sqrt{t^{2n-2} + h^2}}\, dt\ .$$

Note that on $\partial B(x, \delta)$,

$$w(y) - u^h(y) \geq (1 - \sqrt{1 - \frac{1}{C^2}}) \int_0^{\delta} \frac{h}{\sqrt{t^{2n-2} + h^2}}\ dt > 0\ .$$

Hence by the maximum principle, $u^h(y) < w(y)$ in $B(x, \delta)$. In particular at $y = x$,

$$u^h(x) < V_u(x) + (1 - \varepsilon)\sqrt{1 - \frac{1}{C^2}}\ \delta - \sqrt{1 - \frac{1}{C^2}} \int_0^{\delta} \frac{h}{\sqrt{t^{2n-2} + h^2}}\ dt\ .$$

Now let $h \to \infty$ to conclude $V_u(x) \leq V_u(x) - \varepsilon\sqrt{1 - \frac{1}{C^2}}\ \delta$, a contradiction, so the claim is proven and the proposition is completed. ∎

5. Construction of Counterexamples.

We will follow Treiberg's idea (see [12]) to construct counterexamples, more precisely we will construct solution to equation (1.4) such that $u(x) \to \tilde{C}|x| - \frac{n-1}{C^2}\log|x| + f\left(\frac{\tilde{C}x}{|x|}\right)$, as $|x| \to \infty$, where $\tilde{C} = \sqrt{1-\frac{1}{C^2}}$ and $f \in C^2(S^{n-1}_{\tilde{C}})$.

We extend the function f to $\mathbb{R}^n \setminus \{0\}$ by defining $f(\tilde{C}x) = f\left(\frac{\tilde{C}x}{|x|}\right)$. Since $f \in C^2$, we have for all $x, y \in S^{n-1}$:

$$|f(\tilde{C}x) - f(\tilde{C}y) - Df(\tilde{C}y)(\tilde{C}x - \tilde{C}y)| \le M|\tilde{C}x - \tilde{C}y|^2 = -2\tilde{C}My \cdot (\tilde{C}x - \tilde{C}y). \tag{5.1}$$

Let $p_1(\tilde{C}y) = Df(\tilde{C}y) + 2M\tilde{C}y$ and $p_2(\tilde{C}y) = Df(\tilde{C}y) - 2M\tilde{C}y$, so that

$$p_1(\tilde{C}y) \cdot (\tilde{C}x - \tilde{C}y) \le f(\tilde{C}x) - f(\tilde{C}y) \le p_2(\tilde{C}y) \cdot (\tilde{C}x - \tilde{C}y). \tag{5.2}$$

Now let $\psi(x)$ denote the rotationally symmetric solution to (1.4) (see [9]). We know that $\psi(x) \to \tilde{C}|x| - \frac{n-1}{C^2}\log|x| + o(1)$ as $|x| \to \infty$. Let $z_1(x;\tilde{C}y) = f(\tilde{C}y) - p_1(\tilde{C}y) \cdot \tilde{C}y + \psi(x + p_1(\tilde{C}y))$ and $z_2(x;\tilde{C}y) = f(\tilde{C}y) - p_2(\tilde{C}y) \cdot \tilde{C}y + \psi(x + p_2(\tilde{C}y))$. Then by equation (5.2) we have

$$f(\tilde{C}x) \ge z_1(rx;\tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2}\log r \text{ as } r \to \infty,\ x, y \in S^{n-1}, \tag{5.3}$$

and

$$f(\tilde{C}x) \le z_2(rx;\tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2}\log r \text{ as } r \to \infty,\ x, y \in S^{n-1}. \tag{5.4}$$

Therefore,

$$\begin{aligned} &\lim_{r\to\infty} z_1(rx;\tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2}\log r \le f(\tilde{C}x) \\ &\le \lim_{r\to\infty} z_2(rx;\tilde{C}y) - \tilde{C}r + \frac{n-1}{C^2}\log r \end{aligned} \tag{5.5}$$

for $x \in S^{n-1}$.

Let $q_1(x) = \sup_{y\in S^{n-1}} z_1(x;\tilde{C}y)$ and $q_2(x) = \inf_{y\in S^{n-1}} z_2(x;\tilde{C}y)$. Then, $q_1(x) \le q_2(x)$ and $q_i(x)$ (i=1,2) tends to $f(\tilde{C}x) + \tilde{C}r - \frac{n-1}{C^2}\log r$ as $r \to \infty$.

Lemma 5.1. *There exists a smooth solution u to the Dirichlet problem*

$$\begin{cases} a^{ij}u_{ij} - Cw + 1 = 0 \text{ in } G \\ u = 0 \text{ on } \partial G \end{cases} \tag{5.6}$$

where G is a convex $C^{2,\alpha}$ domain in $\mathbb{R}^n$.

Proof. Let $d = \operatorname{diam}(G)$ be the diameter of G. For any $y \in \partial G$, we can choose coordinates such that $y = (y_1, 0, \cdots, 0)$ and $G \subset \{x \mid |x_1| \leq y_1\}$, where $0 < y_1 \leq d/2$. Let $\underline{u}(x) = \tilde{C}x_1 - \tilde{C}y_1$, $\bar{u} \equiv 0$. Then $\underline{u} \leq \bar{u}$ in G and $\underline{u}$ satisfies

$$a^{ij}\underline{u}_{ij} - Cw_{\underline{u}} + 1 = 0 \tag{5.7}$$

By the maximum principle, any solution u to the Dirichlet problem (5.6) satisfies

$$\underline{u} \leq u \leq \bar{u} \, \text{on } G. \tag{5.8}$$

so $|Du(x)| \leq \tilde{C}$ on $\partial\bar{G}$. Combined with equation (2.7) we conclude that

$$|Du(x)| \leq \tilde{C} \text{ on } \bar{G}. \tag{5.9}$$

Now it is standard (see [6]) to prove that a smooth solution $u \in C^{2,\alpha}(\overline{G})$ exists. □

Finally, we will find a sanwiched solution u such that

$$q_1 \leq u < q_2.$$

Let ϕ be a strictly spacelike hypersurface $q_1 \leq \phi < q_2$ so that $\phi(0) = q_1(0)$ and $G_m = \phi^{-1}((-\infty, m))$ is a convex domain with $C^{2,\alpha}$ boundary. By lemma 5.1 we know there is an analytic solution u_m to the Dirichlet problem

$$\begin{aligned} &a^{ij}u_{ij} - Cw + 1 = 0 \text{ on } G_m \\ &u = m \text{ on } \partial G_m. \end{aligned} \tag{5.10}$$

Therefore, we find a sequence of finite solutions u_m with $q_1 \leq u_m < q_2$ defined on convex domains G_m which exhaust $\mathbb{R}^n$.

Next, let K be a compact subset of $\mathbb{R}^n$. Then, by equation (5.9) there are constants $r_1 < r_2$ so that for sufficiently large m we have

$$dist_m(0, x) < r_1, \text{ for all } x \in K,$$

$$dist_m(0, x) < r_2, \text{ for all } x \in \partial G_m,$$

where $dist_m(0, x)$ is the intrinsic distance between the points $(0, u_m(0))$ and $(x, u_m(x))$ on $\Sigma_m = \{(x, u_m(x)|x \in G_m)\}$.

At last, following the proof of Lemma 3.3, we find u_m has uniform C^3 bounds on compact subsets. Hence, a subsequence can be extracted that converges to a global solution of equation (1.4). Moreover, $\lim\limits_{m_j \to \infty} u_{m_j} = u$ satisfies $u(x) \to \tilde{C}|x| - \frac{n-1}{C^2}\log|x| + f(\tilde{C}x)$. Thus we have proved

Theorem 5.2. *Let $f \in C^2(S^{n-1}_{\tilde{C}})$. Then there exists an entire strictly spacelike hypersurface u satisfying equation* (1.4) *such that*

$$u(x) \to \tilde{C}|x| - \frac{n-1}{C^2}\log|x| + f(\tilde{C}x) \text{ as } |x| \to \infty.$$

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