GEOMETRIC ASPECTS OF THE THEORY OF FULLY NON LINEAR ELLIPTIC EQUATIONS

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Lecture 1: Introduction

In these lectures, we will talk about various aspects of the theory of fully nonlinear elliptic equations as they pertain to Global Differential Geometry. Advances in this theory in the last twenty years have opened the possibility of tackling extremely complicated existence questions. However, fully nonlinear elliptic equations arise naturally in many areas of geometry and we will also try to illustrate many diverse applications and tools.

For geometric applications, the most important class of fully nonlinear elliptic equations are implicitly defined equations of the form

$$F(A) = f(\kappa_1, \dots, \kappa_n) = \psi \tag{1.1}$$

where A, for example, is the second fundamental form of a hypersurface, $f(\lambda)$ is a symmetric function of the eigenvalues of A and ψ is a function of position and the unit normal. Thus in the geometric setting we are studying functions of the principle curvatures of a hypersurface S. We will call a surface S an elliptic Weingarten surface if it satisfies $f(\kappa) = \text{constant} > 0$ and this equations in local coordinates is a nonlinear eliptic equation. In recent years, the study of hypersurfaces with higher order curvature,

$$f(\kappa) = H_r(\kappa) = \frac{S_r(\kappa)}{S_r(1, \dots, 1)} = 1$$

has received considerable attention. The linear case (r=1) of mean curvature H is classical while Gauss curvature K (r=n) is the protypical fully nonlinear case. The case r=2 of scalar curvature has been much much less studied but is clearly of great geometric interest. In the next section, we show that hypersurfaces with curvature

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quotient $\frac{H_r(\kappa)}{H_s(\kappa)} = 1$ constant also falls into the general theory.

Before we begin to study the properties of implicity defined elliptic equations, we will first give a few more (nonstandard) examples.

Example 1.1. (Special Lagrangian Graphs) Let Ω be a bounded domain in R^n and consider the graph $\nabla u : \Omega \to R^{2n}$ as an n dimensional submanifold of R^{2n}. The work of Harvey and Lawson [21] shows that such a graph is special Lagrangian (and in particular is absolutely area minimizing) with respect to the standard symplectic structure if u satisfies,

$$f(\lambda) = \Im(\delta_{ij} + iu_{ij}) = \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k S_{2k+1}(\lambda(u_{ij})) = 0 \text{ in } \Omega \quad .$$

This fully nonlinear equation although elliptic does not fit directly into the fully nonlinear theory because of the concavity requirement. However in [5] it is shown that at least one of the level sets of $f(\lambda) = 0$ is concave and this allows the definition of a new $g(\lambda)$ which is elliptic and concave on the admissible class of functions where the eigenvalues of the hessian lie inside this level set. For recent work, see Yuan [40].

Example 1.2. (Conformal equations on S^n) Let (S^n, g_0) be the n sphere with the standard metric and let g be a metric conformal to g_0. Viaclovsky [39], Chang-Gursky-Yang [9] and Li-Li [26] have studied the (global) fully nonlinear equation

$$S_k(R_{ij} - \frac{R}{2(n-1)} g_{ij}) = 1$$

where R_{ij} , R are respectively the Ricci tensor and scalar curvature of the metric g. Let $x = (x_1, \dots , x_n)$ be cooordinates obtained from stereographic projection from the North pole and write

$$g = u(x)^{-2}\ dx^2 = u(x)^{-2}(1 + |x|^2)^2 g_0\ , u > 0\ .$$

Then the above equation is equivalent to

$$S_k(uu_{ij} - \frac{|\nabla u|^2}{2}\delta_{ij}) = 1\ .$$

This equation is conformally invariant, that is, for any conformal transformation $T : R^n \to R^n$,

$$v(x) = |J(x)|^{-\frac{1}{n}} u(Tx)$$

is again a solution.

Example 1.3. (Global problems associated to Curvature flows) Let S_0 be a compact hypersurface of R^n and consider a flow with normal velocity given by a function of curvature $V^N = f(\kappa)$. For example, flow by mean or Gauss curvature, $f = H\ ,\ K$ or perhaps the more exotic harmonic curvature (example of a curvature quotient)

$$H_{n,n-1} = \frac{H_n}{H_{n-1}} = \frac{n}{\sum_{i=1}^{n} \frac{1}{\kappa_i}}$$

(see Huisken-Ilmanen [24]).

Associated to such a flow we have a global problem in Differential Geometry: Let S be a compact hypersurface satisfying $f(\kappa) = X \cdot \nu \geq 0$. Is S necessarily a sphere? This is the problem of asymptotic shape for the flow, at least for S_0 convex. If we start with such an S, then it flows by homethety. The problem is somewhat subtle; if S is an ellipsoid in R^3, then its Gauss curvature always satisfies $K = (X \cdot \nu)^4$ so for $f(\kappa) = K^{\frac{1}{4}}$ ellipsoids are solutions. See Andrews [1, 2].

Another global problem associated to the flow is to characterize translating solitons. These are solutions which are entire graphs $x_{n+1} = u(x)$ satisfying $f(\kappa) = \nu^{n+1}$ Assuming for example that u is convex, is u necessarily radially symmetric with respect to some origin?

For example, if $f = S_1(\kappa)$ then u satisfies

$$(\delta_{ij} - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij}) = 1\ .$$

If u is convex, is u necessarily radial? The difficulty is to establish that u is asymptotically quadratic.

If S is the graph of a function $x_{n+1} = u(x)$, defined in a domain $\Omega \subset R^n$, then the Gauss curvature of the graph may be expressed as

$$K(S) = \frac{\det\ u_{ij}}{(1 + |\nabla u|^2)^{\frac{n+2}{2}}}$$

So if $K(S) = K > 0$ constant and we want to prescribe boundary values, we arrive at the Monge-Ampere boundary value problem

$$\det u_{ij} = K(1+|\nabla u|^2)^{\frac{n+2}{2}} \text{ in } \Omega \tag{1.2}$$

$$u = \phi \text{ on } \partial\Omega\ . \tag{1.3}$$

It is not difficult to see that the Monge-Ampere operator is elliptic only for the admissible class of strictly convex functions, corresponding to the class of hypersurfaces S with principle curvatures which are strictly positive. The general existence and regularity theory for fully nonlinear equations $F(D^2u, Du) = \psi(x, u, Du)$ is well developed for equations which are uniformly elliptic and concave (that is $L = F^{ij}\partial_i\partial_j$ is uniformly elliptic where $F^{ij} = \frac{\partial F}{\partial u_{ij}}$ and F is also concave in D^2u) because of the fundamental Evans-Krylov interior $C^{2+\alpha}$ regularity results and their extensions up to the boundary (see [14] and [4]). In the case of the Monge-Ampere equation, the operator $F(D^2u) = (\det u_{ij})^{\frac{1}{n}}$ is elliptic (but not uniformly elliptic) and concave on the admissible class of strictly convex functions. This reduces existence and higher regularity questions to the question of obtaining $C^{1,1}$ apriori estimates.

In the geometric setting, we associate to the equation $F(A) = f(\kappa_1, \dots, \kappa_n)$ the linearized operator $L = F^{ij}(A)\nabla_i\nabla_j$ where $F^{ij} = \frac{\partial F}{\partial a_{ij}}$ and ∇_i is a covariant derivative. Thus L is elliptic if F^{ij} if (say) positive definite.

We now show that the ellipticity and concavity of the nonlinear operator

$$F(A) = f(\lambda)$$

where the λ are the eigenvalues of a matrix function A, can be completely understood from the properties of $f(\lambda)$ We assume that $f(\lambda)$ is a symmetric function defined in an open convex cone symmetric Γ with vertex at the origin. The symmetry of f implies that if f is a smooth function, then $F(A)$ is also a smooth function.

Theorem 1.4. *Assume that $f_{\lambda_i} > 0\ \forall i$ and that $f(\lambda)$ is concave.*
i. When A is diagonal, $F^{ij} = f_i\ \delta_{ij}$ so $L = F^{ij}\nabla_i\nabla_j$ is elliptic.
ii. $F(A)$ is concave.
iii. When A is diagonal with simple eigenvalues,

$$F^{ij,rs} = f_{ir}\delta_{ij}\delta_{rs} + \frac{f_i - f_j}{\lambda_i - \lambda_j}\ (1-\delta_{ij})\ \delta^{ij}_{rs}\ , \tag{1.4}$$

where $F^{ij,rs} = \frac{\partial^2 F}{\partial a_{ij}\partial a_{rs}}$ and $\delta^{ij}_{rs} = 1$ if $\{ij\} = \{rs\}$ and 0 otherwise.

Proof. . The computation depends on computing the first and second variations of the eigenevalues of a symmetric matrix when it is diagonal. We first give a quick proof of the concavity of $F(A)$ (assuming we have already demonstated part i.) by making use of the following well-known lemma from linear algebra.

Lemma 1.5. *Let (F^{ij}) , (B_{ij}) be symmetric nxn matrices with eigenvalues*

$$\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n \geq 0$$
$$\mu_1 \leq \mu_2 \leq \ldots \mu_n$$

Then $\sum F^{ij} B_{ij} \geq \gamma_1 \mu_1 + \ldots \gamma_n \mu_n$.

Now let A,B be symmetric matrices with $\lambda(A)$, $\lambda(B) \in \Gamma$. To demonstrate the concavity of $F(A)$ we want to show that

$$F^{ij}(A)(B-A)_{ij} \geq F(B) - F(A) \ .$$

Let B have eigenvalues $\mu_1 \leq \mu_2 \leq \ldots \mu_n$ and let A have eigenvalues $\{\lambda_i\}_{i=1}^n$ arranged so that (the eigenvalues of $G^{ij}(A)$)

$$f_{\lambda_1} \geq f_{\lambda_2} \geq \ldots \geq f_{\lambda_n} > 0 \ .$$

By the lemma, $G^{ij} B_{ij} \geq f_{\lambda_1} + \ldots + f_{\lambda_n} \mu_n$ so that

$$F^{ij}(B_{ij} - A_{ij}) \ = \ G^{ij} B_{ij} - \sum \lambda_i f_{\lambda_i} \geq \sum f_{\lambda_i}(\lambda)(\mu_i - \lambda_i) \geq f(\lambda) - f(\mu)$$

by the concavity of f, proving part ii.
To prove part i. we write $F^{ij} = \sum_k f_{\lambda_k} \frac{\partial \lambda_k}{\partial A_{ij}}$. We compute $\frac{\partial \lambda_k}{\partial A_{ij}}$ at a generic point where the eigenvalues are simple by making a variation $\tilde{A}_{ij} = A_{ij} + \epsilon$. If $j < i$ then

$$\det\,(\tilde{A} - \lambda\ I) = \Pi_{k \neq i,j}(\lambda_k - \lambda)(\lambda^2 - (\lambda_i + \lambda_j) + \lambda_i \lambda_j - \epsilon^2\ ).$$

Hence $\tilde{\lambda}_k = \lambda_k$ if $k \neq i, j$ and

$$\tilde{\lambda}_i = \frac{\lambda_i + \lambda_j}{2} + \sqrt{(\frac{\lambda_i - \lambda_j}{2})^2 + \epsilon^2}\ ,$$
$$\tilde{\lambda}_j = \frac{\lambda_i + \lambda_j}{2} - \sqrt{(\frac{\lambda_i - \lambda_j}{2})^2 + \epsilon^2}\ \ .$$

It follows that $\tilde{\lambda}_i = \lambda_i + O(\epsilon^2)$ and similarly $\tilde{\lambda}_j = \lambda_j + O(\epsilon^2)$. Hence $\frac{\partial \lambda_k}{\partial A_{ij}} = 0$ for $k \neq i, j$. If now $i = j$ then $\tilde{\lambda}_k = \lambda_k$ if $k \neq i$ and $\tilde{\lambda}_i = \lambda_i + \epsilon$. Hence in all cases,

$\frac{\partial \lambda_k}{\partial A_{ij}} = \delta_{ki}\delta_{ij}$. It follows that $F^{ij} = \sum_k f_{\lambda_k} \frac{\partial \lambda_k}{\partial A_{ij}} = f_{\lambda_i}\delta_{ij}$, proving part i. To prove part iii. we use

$$F^{ij,rs} = \sum_k f_k \frac{\partial^2 \lambda_k}{\partial A_{ij}\partial A_{rs}} + \sum_{k,l} f_{kl} \frac{\partial \lambda_k}{\partial A_{ij}} \frac{\partial \lambda_l}{\partial A_{rs}} \ .$$

¿From our previous calculations we see that

$$\sum_{k,l} f_{kl} \frac{\partial \lambda_k}{\partial A_{ij}} \frac{\partial \lambda_l}{\partial A_{rs}} = f_{ir}\delta_{ij}\delta_{rs} \ .$$

Similarly, we see that $\frac{\partial^2 \lambda_k}{\partial A_{ij}\partial A_{rs}}$ is nonzero only when $\{i,j\} = \{r,s\}$ and $i \neq j$ and

$$\frac{\partial^2 \lambda_i}{\partial A_{ij}^2} = \frac{1}{\lambda_i - \lambda_j} \ , \ \frac{\partial^2 \lambda_j}{\partial A_{ij}^2} = -\frac{1}{\lambda_i - \lambda_j} \text{ if } \lambda_i > \lambda_j \ .$$

The formula follows this. Note also that $\frac{f_i - f_j}{\lambda_i - \lambda_j} \leq 0$. To see this, suppose $\lambda_i > \lambda_j$ and note that by the symmetry of the convex cone Γ, the ray $\lambda^* + t(\lambda_i - \lambda_j)(e_i - e_j)$, $0 \leq t \leq 1$, is in Γ, where λ^* is obtained from λ by interchanging λ_i and λ_j. Since f is symmetric and concave, the graph of $t \to f(\lambda^* + t(\lambda_i - \lambda_j)(e_i - e_j))$ is concave and symmetric about its maximum which occurs at $t = \frac{1}{2}$. Hence

$$(\lambda_i - \lambda_j)(f_i(\lambda) - f_j(\lambda)) \leq 0$$

from which the result follows. □

As a nice application of the theorem, we can give purely algebraic necessary and sufficient conditions on $f(\kappa)$ which guarantee that the linearized operator is a divergence operator.

Theorem 1.6. *$L = F^{ij}\nabla_i\nabla_j$ is of divergence form if and only if*

$$f_{jj} = 0 \text{ and } \sum_{r \neq j} (f_{rj} + \frac{f_r - f_j}{\lambda_r - \lambda_j}) = 0 \ \forall j \ . \tag{1.5}$$

Proof. We use $\nabla_k A_{ij} = \nabla_i A_{kj}$ (Codazzi equations) for a hypersurface in R^{n+1} with second fundamental form A_{ij}. Hence $L = \nabla_i(F^{ij}\nabla_j) - F^{ij,rs}\nabla_i A_{rs}$ is a divergence if and only if $F^{ij,rs}\nabla_i A_{rs} = 0$. We may suppose that A_{ij} is diagonal with simple eigenvalues; then

$$\begin{aligned} F^{ij,rs}\nabla_i A_{rs} &= \sum_r f_{jr}\nabla_j A_{rr} + \sum_{r \neq j} \frac{f_r - f_j}{\lambda_r - \lambda_j} \nabla_r A_{rj} \\ &= f_{jj}\nabla_j A_{jj} + \sum_{r \neq j} (f_{rj} + \frac{f_r - f_j}{\lambda_r - \lambda_j}) \nabla_r A_{rj} \ . \end{aligned}$$

Since this must hold for all possible values of $\nabla_j A_{jj}$, $\nabla_r A_{rj}$ the result follows. □

It is well-known (see for example [31]) that the linearized operator for the higher order mean curvatures are of divergence form. This is usually proven through the use of the so called Newton tensors. Instead, we give a simple proof using Theorem 1.6.

Corollary 1.7. *$f = S_r$ satisfies (1.5) and so its associated operator L_r is of divergence form.*

Proof. Let $f = S_r(\kappa)$; then (see Lemma 2.14 below) $f_i = S_{m-1}(\kappa'_i)$, and so $f_r = S_{m-1}(\kappa'_r) = \kappa_j S_{m-2}(\kappa'_{rj}) + S_{m-1}(\kappa'_{rj})$. Therefore,

$$\sum_{r\neq j} \frac{f_r - f_j}{\kappa_r - \kappa_j} = -\sum_{r\neq j} S_{m-2}(\kappa'_{rj}) \ .$$

On the other hand,

$$f_{jj} = 0 \text{ and } \sum_{r\neq j} f_{rj} = \sum_{r\neq j} S_{m-2}(\kappa'_{rj}) = \sum_{r\neq j} \frac{f_r - f_j}{\kappa_r - \kappa_j}$$

as required. □

An outline of the content of the next three lectures is as follows. In Lecture 2, we present many basic results (concavity, ellipticity) on elementary symmetric functions and curvature quotients that are used. The presentation is essentially self-contained. In Lecture 3, we derive the fundamental identities on Weingarten surfaces extending the well known method for minimal surfaces. As an application we present a new variant [10] of Alexandrov reflection, the "method of moving spheres". In Lecture 4, we discuss the Monge-Ampere equation and applications to the existence of hypersurfaces of constant positive Gauss curvature (K-hypersurfaces for short) [18] [34]. For very recent work on the existence of immersed K-hypersurfaces in R^{n+1} see [19][38].

Lecture 2. Hyperbolic Polynomials, elementary symmetric functions and convexity

Garding's work on hyperbolic polynomials (see [11], [23]) is important to the study of curvature functions.

Definition 2.1. A homogeneous polynomial $p(x)$ of degree m in R^n is said to be hyperbolic with respect to $a \in R^n$ (abbreviated hyp(a)) if the equation $p(x+ta)=0$ of degree m (in t) has exactly m real roots for every $x \in R^n$.

Necessarily $p(a) \neq 0$ and we will assume $p(a) > 0$. It is also easy to see that $p(x)$ has real coefficients and that $Q = \sum_j a_j \frac{\partial}{\partial \lambda_j} P$ is hyp(a) by Rolle's theorem .

Theorem 2.2. *Let $p(x)$ be hyp(a). Then the component Γ of $\{x \in R^n \ : \ p(x) \neq 0\}$ containing a is a convex cone, p(x) is hyp(b) for any $y \in \Gamma$ and moreover, $(p(x))^{\frac{1}{m}}$ is concave .*

Proof. For simplicity assume $p(a)=1$ (as we will in applications). Set

$$\Gamma_a = \{x \in R^n \ : \ p(x+ta) \neq 0 \ , \ t \geq 0\}$$

Since $p(x+ta) = p(a)\Pi(t-t_i(x))$, $x \in \Gamma_a$ if and only if all the t_i are negative. Now Γ_a is open and $a \in \gamma_a$ since $p(a+ta) = (1+t)^m$ has only the real root -1. Suppose $x \in \overline{\Gamma_a}$. Then $p(x+ta) \neq 0$ for $t > 0$ so $x \in \Gamma_a$ if $p(x) \neq 0$. Hence Γ_a is open and closed in $\{x|p(x) \neq 0\}$. But Γ_a is starshaped with respect to a ($x \in \Gamma_a \Rightarrow \alpha x \ , \ \alpha x + \beta a \in \Gamma_a$ for $\alpha \ , \ \beta > 0$) so $\Gamma_a = \Gamma$. ☐

For $y \in \Gamma$ and $\varepsilon > 0$ fixed, set

$$E_{y,\ \varepsilon} = \{x \in R^n \ : \ p(x + i\varepsilon \ a + is \ y) \neq 0 \ , \ \Re s \geq 0\}$$

The $E_{y,\ \varepsilon}$ is open and contains 0 since $p(i\varepsilon \ a + is \ y) = (is)^m \ p(\varepsilon \frac{a}{s} + y) = 0$ implies $s < 0$ (since $y \in \Gamma = \Gamma_a$). If $x \in \overline{E_{y,\ \varepsilon}}$, then $p(x + i\varepsilon \ a + is \ y) \neq 0$ by a theorem of Hurwitz if $\Re s > 0$, while if $\Re s = 0$, $z = x + is \ y \in R^n$, $p(z + i\varepsilon \ a) = \Pi(i\varepsilon - t_i(z)) \neq 0$. Hence $E_{y,\ \varepsilon}$ is both open and closed, so $E_{y,\ \varepsilon} = R^n$.

In particular (put $s=1$), we have shown

$$p(x + i(\varepsilon \ a + y) \neq 0 \ , \ \text{if } x \in R^n \ , \ y \in \Gamma \ , \ \varepsilon > 0 \ .$$

Since Γ is open, this remains true for $\varepsilon = 0$, i.e , $p(x+iy) \neq 0$ for all $x \in R^n$. It follows that the equation $p(x+ty) = 0$ has only real roots for if $t = t_1 + it_2$ is a root with $t_2 \neq 0$, this would mean $p((\frac{x+t_1 y}{t_2}) + iy) = 0$, a contradiction. Thus we have shown that $p(x)$ is hyp(y) for any $y \in \Gamma$. Hence y can play the role of a so Γ is starshaped with respect to every point in Γ so is convex.

Finally, we will prove the concavity of $(p(x))^{\frac{1}{m}}$ and the strict monotonicity of $p(x)$ in each argument.

Proposition 2.3. *i. For any $y \in \Gamma$ and $x \in R^n$, the function $\phi(s) = (p(sx+y))^{\frac{1}{m}}$ is concave in s when $sx + y \in \Gamma$. In particular, take $x = z - y$ with $z \in \Gamma$ and $0 \leq s \leq 1$; then*

$$(p(sz + (1-s)y))^{\frac{1}{m}} \geq s(p(z))^{\frac{1}{m}} + (1-s)(p(y))^{\frac{1}{m}}$$

ii. $p_{x_i} > 0$ in Γ.

Proof. $p(x + ty) = p(y)\Pi(t - t_i)$, so

$$p(sx + y) = s^m \; p(x + \frac{y}{s}) = p(y)\Pi(1 - st_i) \; .$$

Since $sx + y \in \Gamma$, $1 - st_i > 0$ for all i. Set $f(s) = \log \phi(s)$; then

$$f'(s) = -\sum \frac{t_i}{1 - st_i} \; , \; f''(s) = -\sum \frac{t_i^2}{(1 - st_i)^2} \; .$$

Hence,

$$m^2 e^{-\frac{f(s)}{m}} \frac{d^2}{ds^2} e^{\frac{f(s)}{m}} = f'(s)^2 + mf''(s)$$

$$= (\sum(\frac{t_i}{1 - st_i})^2 - m\sum \frac{t_i^2}{(1 - st_i)^2} \leq 0$$

by Cauchy-Schwartz, completing the proof. ☐

Corollary 2.4. *Let $p(x)$be a symmetric hyperbolic polynomial of degree m with respect to $a = (1, 1, \dots, 1)$ Then,*
i. Γ is an open (proper)convex cone containing the positive cone Γ_+ and contained in the half-space $\sum x_i > 0$.
ii. $f(x) = p(x)^{\frac{1}{r}}$ is concave and $f_{x_i} > 0 \; \forall i$.

The last condition is the ellipticity (we explained this in the introduction) and follows easily from the concavity, positivity and analyticity of $f(x)$ in Γ. Starting with $S_n(x) = \Pi_{i=1}^n x_i$, $a = (1, \dots, 1)$ which is obviously hyperbolic, we obtain successively by applying the differential operator $\sum_j \frac{\partial}{\partial x_j}$ that all the elementary symmetric functions $S_r = \sum_{i_1 < \dots < i_r} x_{i_1} \dots x_{i_r}$ are also hyp(a) and that $\Gamma(S_r) \subset \Gamma(S_{r-1})$.

Thus the elementary symmetric functions $S_r(\kappa)$ of curvature are the most important examples for study. It is useful to normalize S_r and make the

Definition 2.5. Set $H_r(x) = \frac{S_r(x)}{\binom{n}{r}}$. The curvature functions $H_r(\kappa)$ of a hypersurface are called the higher order mean curvatures with

$$H_1(\kappa) = \frac{1}{n} \sum \kappa_i \text{ is the mean curvature.}$$

$$H_2(\kappa) = \frac{2}{n(n-1)} \sum_{i<j} \kappa_i \kappa_j \text{ is the scalar curvature .}$$

$$H_n(\kappa) = \Pi k_i \text{ is the Gauss curvature .}$$

The S_r and H_r and the cones Γ_r are surprisingly complex and we will will need additional properties. Besides being interesting and important in their own right, they will allow us to introduce other curvature functions that are worthy of further study.

We first state the following elementary but useful lemma which follow from repeated applications of Rolle's theorem.

Lemma 2.6. *([20])Let $P(x,y)$ be a homogeneous polynomial in x and y with real coefficients such that all of its roots $\frac{x}{y}$ are real. Then all polynomials derived by differentiation with repect to x and y also have only real roots.*

Let's apply this to

$$P(x,y) = \Pi(x + \lambda_i\ y) = \sum_{r=0}^{n} S_r(\lambda)\ x^{n-r}\ y^r\ .$$

We obtain

$$\frac{\partial^{n-2} P}{\partial^{n-r-2} x\ \partial^r y} = \frac{n!}{2}(H_r(\lambda)\ x^2 + 2H_{r+1}(\lambda)\ xy + H_{r+2}(\lambda) y^2)$$

Hence the discriminant must be nonegative and we have shown

Proposition 2.7. *(Newton inequalities) For any* $\lambda \in R^n$*,*

$$H_{r-1}H_{r+1} < H_r^2 \ , \ 1 \le r \le n$$

with equality only if all the components of λ *are equal.*

Corollary 2.8. *(Maclaurin inequalities) If* $\lambda \in \Gamma_r$ *, then*

$$H_r^{\frac{1}{r}} \le H_{r-1}^{\frac{1}{r-1}}$$

Proof. We have

$$(H_0 H_2)(H_1 H_3)^2 \cdots (H_{r-2}H_r)^r \le H_1^2 H_2^4 \cdots H_{r-1}^{2r}$$

so $H_r^{r-1} \le H_{r-1}^r$. □

We now consider another interesting class of curvature functions.

Definition 2.9. For $\lambda \in \Gamma_r$ and $r > s$ define the curvature quotient

$$H_{r,s}(\lambda) = \frac{H_r(\lambda)}{H_s(\lambda)} \ .$$

Using the Newton inequalities, we can also prove

Lemma 2.10. *(generalized Newton-Maclaurin inequalities) If* $\lambda \in \Gamma_r$ *,* $0 \le s < r$ *and* $0 \le l < k$*, then*

$$(H_{r,s})^{\frac{1}{r-s}}) \le (H_{k,l})^{\frac{1}{k-l}}$$

provided $r \ge k$ *and* $s \ge l$ *.*

Proof. Note that the Newton inequalities may be expressed as

$$H_{i+1,i} \le H_{i,i-1}$$

for $1 \le i \le r-1$. We proceed by induction on r starting from the trivial case r=1. Then for $k \le r$,

$$(H_{r+1,s})^{\frac{1}{r-s}} = (H_{r+1,r}H_{r,s})^{\frac{1}{r+l-s}} \le (H_{k,l})^{\frac{1}{k-l}}(H_{k,l})^{\frac{r-s}{k-l}})^{\frac{1}{r+1-s}} = H_{k,l}^{\frac{1}{k-l}}$$

completing the induction. On the other hand if $k = r+1$ then

$$(H_{r+1,s})^{\frac{1}{r+1-s}} \le (H_{r+1,l})^{\frac{1}{r+1-l}}$$

is equivalent to

$$(H_{r+1,l})^{\frac{1}{r+1-l}}) \le (H_{s,l})^{\frac{1}{s-l}} \ .$$

Since $s \leq r$ the previous induction applies and the proof is complete □

We next prove the crucial concavity of $H_{r,s}$.

Theorem 2.11. $(H_{r,s})^{\frac{1}{r-s}}$ *is concave on* Γ_r.

We reduce to the case $s = r-1$ using that

$$H_{r,s} = \Pi_{i=s+1}^{r} H_{i,i-1}$$

is a product of $r - s \geq 1$ positive terms of homogeneity 1. Suppose we have proved that $H_{i,i-1}$ is concave. Then the concavity of $H_{r,s}$ follows from the following useful lemma of independent interest.

Lemma 2.12. *Let* $R_i(x)$, $i = 1, \dots, N$ *be positive and concave on a convex cone* Γ *in* R^n*. Then* $R = (\Pi_{i=1}^{N} R_i)^{\frac{1}{N}}$ *is concave on* Γ.

Proof. By the concavity of each R_i,

$$2^N \Pi_{i=1}^{N} R_i(\frac{x_1 + x_2}{2} \geq \Pi_{i=1}^{N}(R_i(x_1) + R_i(x_2)) \tag{2.1}$$

Write $a_i = R_i(x_1) > 0$, $b_i = R_i(x_2) > 0$, $r_i = \frac{a_i}{b_i}$, $r = (r_1, \dots, r_N)$. Then using the Maclaurin inequalities in the positive cone of R^N gives

$$\Pi_{i=1}^{N}(r_i + 1) = \sum_{i=0}^{N} \begin{pmatrix} N \\ i \end{pmatrix} H_i(r) \geq \sum_{i=1}^{N}(1 + \begin{pmatrix} N \\ i \end{pmatrix} (\Pi_1^N r_j)^{\frac{i}{N}}) = (1 + (\Pi_1^N r_j)^{\frac{1}{N}})^N$$

In terms of the variables a_i , b_i we have shown

$$\Pi(a_i + b_i) \geq ((\Pi a_i)^{\frac{1}{N}} + (\Pi b_i)^{\frac{1}{N}})^N$$

Remark 2.13. This is just the well known result that the nth root of the determinant function on positive definite matrices is concave. More generally, we will soon show that if $f(\lambda)$ is concave on the convex cone Γ and the λ_i are the eigenvalues of a matrix A, then $F(A) = f(\lambda)$ is a concave function of A. We just demonstrated this for $f = S_n(\lambda)^{\frac{1}{n}}$.

Recalling equation (2.1) and the definition of a_i , b_i, this is the concavity of R. □

Thus in order to prove the theorem, it remains to show the concavity of $H_{i,i-1}$ or equivalently of $S_{i,i-1} = \frac{S_i}{S_{i-1}}$. In order to do so we need (some of) the following elementary identities which we collect for reference in the

Lemma 2.14. *Write $x_i' \in R^{n-1}$ for the vector obtained from $x \in R^n$ by omitting the ith coordinate. Then,*

$$S_r(x) = x_i \ S_{r-1}(x_i') + S_r(x_i') \tag{2.2}$$

$$rS_r(x) = \sum_{i=1}^{n} x_i S_{r-1}(x_i') \tag{2.3}$$

$$(n-r)S_r(x) = \sum_{i=1}^{n} S_r(x_i') \tag{2.4}$$

$$rS_r(x) = S_1 \ S_{r-1} - \sum_{i=1}^{n} x_i^2 \ S_{r-2}(x_i') \ . \tag{2.5}$$

Using the lemma, we find

$$\frac{\lambda_i \ S_{r-1}(\lambda_i')}{S_{r-1}(\lambda)} = \frac{\lambda_i \ S_{r-1}(\lambda_i')}{\lambda_i \ S_{r-2}(\lambda_i') + S_{r-1}(\lambda_i')} = \frac{\lambda_i S_{r-1,r-2}(\lambda_i')}{\lambda_i + S_{r-1,r-2}(\lambda_i')} \ .$$

Hence

$$S_1(\lambda) - rS_{r,r-1}(\lambda) = \sum_{i=1}^{n} \frac{\lambda_i^2}{\lambda_i + S_{r-1,r-2}(\lambda_i')} \ . \tag{2.6}$$

Since $S_{1,0} = S_1$ is concave, we can start an induction. Assume we have shown that $S_{i,i-1}$ is concave for $1 \leq i \leq r-1$. Restricting to $\{\lambda_i = 0\} \cap \Gamma$, we have that $h(\lambda_i') = S_{r-1,r-2}(\lambda_i')$ is concave and we complete the induction using the following

Proposition 2.15. *Let $\phi^i(\lambda) = \frac{\lambda_i^2}{\lambda_i + h(\lambda_i')}$, $1 \leq i \leq n$ where h is concave and $\lambda_n + h$ is positive. Then ϕ^i is convex in Γ.*

Proof. It suffices to consider the case i=n, and write $\phi = \phi^n$. Then by a straightforward compution, we find that $(D^2\phi) = (\phi_{ij})$ is given by

$$\phi_{\alpha\beta} = -\frac{\lambda_n^2}{(\lambda_n + h)^2}(h_{\alpha\beta} - \frac{2}{\lambda_n + h} h_\alpha h_\beta) \ \alpha, \beta < n \ ,$$

$$\phi_{\alpha n} = -\frac{\lambda_n h}{(\lambda_n + h)^3} h_\alpha \ \alpha < n \ ,$$

$$\phi_{nn} = \frac{2h^2}{(\lambda_n + h)^3}$$

and

$$\phi_{ij}\xi_i\xi_j == \frac{2}{(\lambda_n+h)^3}(h\xi_n - \lambda_n \sum_{\alpha<n} \xi_\alpha h_\alpha)^2 \ .$$

Therefore $D^2\phi$ is positive semidefinite so ϕ is convex. ☐

Theorem 2.16. *Let* $f(\lambda) = (H_{r,s}(\lambda)^{\frac{1}{r-s}}$ *for* $1 \le s < r \le n$ *and* $\lambda \in \Gamma_r$. *Then* $f_{\lambda_i} > 0 \ \forall i$ *and* f *is concave in* Γ.

Proof. It remains only to show the strict monotonicity of f in each variable. This is equivalent to showing that $S_{r,s}$ is strictly monotone in each variable. To simplify the notation, we write $S_k(\lambda'_i) = S_{k;i}$.

$$\begin{aligned}
\frac{\partial S_{r,s}}{\partial \lambda_i} &= \frac{S_s S_{r-1;i} - S_r S_{s-1;i}}{S_s^2} \\
&= \frac{(S_{s;i} + \lambda_i \ S_{s-1;i})S_{r-1;i} - (S_{r;i} + \lambda_i \ S_{r-1;i})S_{s-1;i}}{S_s^2} \\
&= \frac{S_{s;i}S_{r-1;i} - S_{r;i}S_{s-1;i}}{S_s^2} \\
&\ge \frac{n(r-s)}{r(n-s)}\frac{S_{s;i}S_{r-1;i}}{S_s^2} > 0
\end{aligned}$$

where we have used Lemma 2.14 and the generalized Newton- Maclaurin inequalities for the last step. ☐

Lecture 3. Calculus on the hypersurface S , fundamental identities

The curvature function $f(\kappa)$ implicitly defines a nonlinear function $G(b_{ij})$ of the second fundamental form by the relation $G(b_{ij}) = f(\kappa)$. Therefore, we can apply the theory of such operators developed in [5]. Naturally associated to this operator is the linear elliptic differential operator

$$L = \sum_{i,j=1}^{n} G^{ij} \nabla_i \nabla_j$$

where $G = G(b_i j)$, $G^{ij} = \frac{\partial}{\partial b_{ij}}$. For example, if $f(\kappa) = \sum \kappa_i$, then $L = \Delta$ as is well known.

Instead of working intrinsically using covariant differentiation on S , it is often much easier to

$$\delta = \nabla - \nu\, (\nu \cdot \nabla)$$

denote the tangential gradient operator on S, where ∇ is the gradient operator in R^{n+1}. Let $e_1, \cdots, e_{n+1}$ denote the orthonormal coordinate frame of R^{n+1}, and set

$$\delta_i = e_i \cdot \delta, \;\; \nu^i = e_i \cdot \nu, \;\; 1 \le i \le n+1.$$

Lemma 3.1. [27] *The curvature matrix $[\delta_i \nu^j]$ is symmetric with eigenvalues $(-\kappa_1, \cdots, -\kappa_n, 0)$ on S, where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of S. Moreover, we also have the important commutator formula*

$$\delta_i \delta_j - \delta_j \delta_i = (\nu^i \delta_j \nu - \nu^j \delta_i \nu) \cdot \delta \;. \tag{3.1}$$

Following [17], we extend the symmetric function f to some open, symmetric subset of R^{n+1} in a canonical way. By a local version (see [15, p.108]) of a theorem of Glaeser , we can find a smooth function h which satisfies

$$f(\kappa) = h(\sigma^{(1)}(\kappa), \cdots, \sigma^{(n)}(\kappa)), \;\; \text{for } \kappa \in \Gamma.$$

On the other hand, the elementary symmetric functions $\sigma^{(1)}, \cdots, \sigma^{(n)}$ are naturally defined on R^{n+1}. Set

$$\tilde{\Gamma} = \{\lambda \in R^{n+1} \,:\, (\sigma^{(1)}(\lambda), \cdots, \sigma^{(n)}(\lambda)) \in R^n \quad \text{is in the domain of } h\}.$$

Then $\tilde{\Gamma}$ is a symmetric open subset of R^{n+1}. Now define a smooth symmetric function $\tilde{f}$ on $\tilde{\Gamma}$ by

$$\tilde{f}(\lambda) = h(\sigma^{(1)}(\lambda), \cdots, \sigma^{(n)}(\lambda)), \;\; \text{for } \lambda \in \tilde{\Gamma}.$$

We note that $\Gamma \times \{0\} \subset \tilde{\Gamma}$, and

$$\tilde{f}(\kappa, 0) = f(\kappa), \quad \text{for } \kappa \in \Gamma.$$

It is obvious that at a point $(\kappa, 0) \in \Gamma \times \{0\}$

$$\tilde{f}_i(\kappa, 0) = f_i(\kappa) \ \text{ for } 1 \leq i \leq n.$$

Let G denote the function on the linear space of real $(n+1) \times (n+1)$ symmetric matrices given by

$$G(A) = \tilde{f}(\lambda_1, \cdots, \lambda_{n+1}),$$

where $\lambda_1, \cdots, \lambda_{n+1}$ are the eigenvalues of the symmetric matrix A. Then we may write

$$G(-\delta\nu) = f(\kappa). \tag{3.2}$$

Note that if $\tilde{e}_1, \cdots, \tilde{e}_{n+1}$ form another orthonormal coordinate frame of R^{n+1} and if we set

$$\tilde{\nu}^i = \nu \cdot \tilde{e}_i, \quad \tilde{\delta}_i = \tilde{e}_i \cdot \delta, \quad 1 \leq i \leq n+1,$$

then

$$\nu^i = (e_i \cdot \tilde{e}_j)\tilde{\nu}^j, \ \delta_i = (e_i \cdot \tilde{e}_j)\tilde{\delta}_j,$$

$$\delta_i \nu^j = (e_i \cdot \tilde{e}_k)(e_j \cdot \tilde{e}_l)\tilde{\delta}_k \tilde{\nu}^l,$$

and

$$G^{ij}(-[\delta_k \nu^l]) = G^{ms}([-\tilde{\delta}_k \tilde{\nu}^l])(e_i \cdot \tilde{e}_m)(e_j \cdot \tilde{e}_s), \tag{3.3}$$

where

$$G^{ij}(A) = \frac{\partial G}{\partial A_{ij}}(A), \quad A = [A_{ij}].$$

In particular, if we choose at a fixed point on the graph S, $\tilde{e}_{n+1} = \nu$ (which implies $\tilde{\delta}_{n+1} = 0$) and $\tilde{e}_1, \cdots, \tilde{e}_n$ such that the matrix $[\tilde{\delta}_k \tilde{\nu}^l] = [-\kappa_1, \cdots, -\kappa_n, 0]$ is diagonal, then as verified in [5],

$$G^{ij}([\tilde{\delta}_k \tilde{\nu}^l]) = f_i \ \delta_{ij} \tag{3.4}$$

(here δ_{ij} is the standard Kronecker symbol and is not related to the tangential operators δ_i , δ_j). We also note that

$$\tilde{\nu}^i = 0, \quad 1 \leq i \leq n, \quad \tilde{\nu}^{n+1} = 1;$$
$$\nu^i = e_i \cdot \tilde{e}_{n+1}, \quad 1 \leq i \leq n+1.$$

¿From the above follow the formulae, it is not difficult to verify the

Proposition 3.2.

$$G^{ij}\nu^j\delta_i = 0 \tag{3.5}$$

$$G^{ij}(\delta_{ij} - \nu^i\nu^j) = \sum f_i \tag{3.6}$$

$$G^{ij}\delta_i\nu^j = -\sum \kappa_i f_i \tag{3.7}$$

$$G^{ij}\delta_i\nu^k\delta_j\nu^k = \sum \kappa_i^2 f_i. \tag{3.8}$$

We now can define the elliptic operator on S naturally associated to $f(\kappa)$ by

$$L = G^{ij}\delta_i\delta_j \tag{3.9}$$

Note that by the commutator formula (3.1)) and formula (3.5) above,

$$(G^{ij}\delta_i\delta_j - G^{ij}\delta_j\delta_i) = G^{ij}(\nu^i\delta_j\nu - \nu^j\delta_i\nu)\cdot\delta = 0$$

Remark 3.3. Set $\overline{e_i} = e_i - \nu^i\nu$, the projection of e_i on the tangent space to S at a point $p \in S$. Then $\delta_i = \nabla_{\overline{e_i}}$ is the covariant derivative with respect to the tangent vector $\overline{e_i}$ at p for $i = 1, \dots, n$, and $\delta_{n+1} = 0$. Note also that $\delta_i\nu^j = 0$ and $G^{ij} = 0$ if i or j is $n+1$. In particular, if at $p \in S$ we choose $e_1, \dots, e_n$ to be an orthonormal basis of the tangent space and $e_{n+1} = \nu$ then

$$L = \sum_{i,j=1}^{n} G^{ij}\nabla_i\nabla_j$$

where $G = G(b_{ij})$, $G^{ij} = \frac{\partial}{\partial b_{ij}}$, and b_{ij} is the second fundamental form of S.

We are now ready for the main result of the section.

Theorem 3.4. *(Fundamental identities) For any hypersurface S with position vector X and unit normal ν,*

$$LX = \left(\textstyle\sum_{i=1}^n \kappa_i f_i\right)\nu \tag{3.10}$$

$$L\nu + \sum \kappa_i^2 f_i\nu = -\delta f \tag{3.11}$$

$$LS_1 + \sum k_i^2 f_i \; S_1 = \sum \kappa_i f_i \; |A|^2 - G^{ij,rs} \; (\delta_k\delta_r\nu^s)(\delta_k\delta_i\nu^j) + \Delta f \tag{3.12}$$

Proof. We compute

$$\begin{aligned} LX &= G^{ij}\delta_i(\delta_j k - \nu^j\nu^k)e_k \\ &= -(G^{ij}\delta_i\nu^j)\nu - (G^{ij}\nu^j\delta_i)\nu \\ &= \sum \kappa_i f_i \end{aligned}$$

proving (3.10).

To prove (3.11), we differentiate (3.2) on S with respect to δ_k and use the commutator formula and Proposition 3.2 to obtain

$$-\delta_k f = G^{ij}\delta_k\delta_i\nu^j = G^{ij}\delta_i\delta_k\nu^j + G^{ij}(\nu^k\delta_i\nu^r - \nu^i\delta_k\nu^r)\delta_r\nu^j \tag{3.13}$$

$$= G^{ij}\delta_i\delta_j\nu^k + G^{ij}\delta_i\nu^r\delta_j\nu^r\nu^k \tag{3.14}$$

Finally, to prove (3.12) we differentiate (3.11) respect to δ_k and use the commutator formula several times.

$$-\Delta f = G^{ij}\delta_k\delta_i\delta_j\nu^k - G^{ij,rs}(\delta_k\delta_r\nu^s)(\delta_k\delta_i\nu^j) - S_1\sum\kappa_i^2 f_i \tag{3.15}$$

Now,

$$\begin{aligned} \delta_k\delta_i(\delta_j\nu^k) &= \delta_i\delta_k\delta_j\nu^k + (\nu^k\delta_i\nu^r - \nu^i\delta_k\nu^r)(\delta_r\delta_j\nu^k) \\ \delta_k\delta_j\nu^k &= \delta_j\delta_k\nu^k + (\nu^k\delta_j\nu^r - \nu^j\delta_k\nu^r)\delta_r\nu^k = \delta_j\delta_k\nu^k - \nu^j\delta_k\nu^r\delta_r\nu^k \\ \delta_i\delta_k\delta_j\nu^k &= -\delta_i\delta_j S_1 - \delta_i\{\nu^j\delta_k\nu^r\delta_r\nu^k\} \end{aligned}$$

Using $G^{ij}\nu^i\delta_j = 0$ and $\nu^k\delta_k = 0$ and Remark 3.3, we obtain

$$G^{ij}\delta_k\delta_i\delta_j\nu^k = G^{ij}\delta_i\delta_k\delta_j\nu^k + S_1\sum{\kappa_i}^2 f_i \tag{3.16}$$

$$G^{ij}\delta_i\delta_k\delta_j\nu^k = -LS_1 + \sum\kappa_i f_i\ |A|^2 \tag{3.17}$$

Combining (3.15),(3.16)and (3.17) gives (3.12) and completes the proof of Theorem 3.4.

Corollary 3.5. *Let S be a graph with with respect to the e_{n+1} direction (so that $\nu^{n+1} > 0$) satisfying $f(\kappa) = c > 0$. Then $h = \frac{S_1}{\nu^{n+1}}$ satisfies*

$$Lh + 2G^{ij}\delta_i(\log\nu^{n+1})\delta_j h \geq 0\ . \tag{3.18}$$

In particular, h achieves its maximum on the boundary.

The proof is an elementary computation using Theorem 3.4 which we leave to the reader.

Corollary 3.6. *(Minkowski integral formulas) Let S be a compact embeddded hypersurface and let $f = H_r(\kappa)$ with associated linearized operator $L = L_r$. Then $L_r(\frac{1}{2}|X|^2) = r(H_r(\kappa)\ X \cdot \nu + H_{r-1}(\kappa))$. In particular,*

$$\int_S (H_r(\kappa)\ X \cdot \nu + H_{r-1}(\kappa))\ dA = 0 \tag{3.19}$$

Proof. Note that $L(|X|^2) = 2(rf(\kappa)\ X \cdot \nu + \sum f_i) = 2r(H_r(\kappa)\ X \cdot \nu + H_{r-1}(\kappa)$ by Lemma 2.15. According to Corollary1.7, L_r is divergence free so formula (3.19) follows by integration (no ellipticity of f is needed or assumed).

3.1. **The Method of Moving Spheres for elliptic Weingarten Surfaces.** As an application of the preceeding calculus, we will outline some recent work [10] which develops for Weingarten hypersurfaces "the method of moving spheres" , a variant of Alexandrov's famous method of moving planes, where reflection in a family of planes is replaced by inversion in a family of spheres. The method of moving spheres in the geometric setting was discovered by McCuan [29],[28] for surfaces of constant mean curvature. Here we will extend McCuan's work to a large class of elliptic Weingarten surfaces S, defined by the relation $f(\kappa) = c$. The fact that this is remotely possible is surprising since the inverted surface no longer satisfies a nice equation. Nevertheless, we shall see that because of Theorem 3.4 , enough structure is preserved.

As discussed previously, the function f is a smooth positive symmetric function, positive homogeneous of degree one, defined in an open convex symmetric cone $\Gamma \subset R^n$, with vertex at the origin, and containing the positive cone $\Gamma^+ \equiv \{\kappa \in R^n :$ all $\kappa_i > 0\}$. The hypersurface S is assumed to be "elliptic", i.e.

$$f_i \equiv \frac{\partial f}{\partial \kappa_i} > 0 \text{ in } \Gamma, \text{ for } 1 \leq i \leq n\ , \tag{3.20}$$

and we assume that f is concave in Γ.

Without loss of generality, we will assume that f is normalized by the condition

$$f(1, \dots, 1) = 1 \tag{3.21}$$

We will also need the technical assumption

$$\sum {\kappa_i}^2 f_i \geq f(\kappa)^2\ . \tag{3.22}$$

We next show that the curvature quotients $f(\kappa) = H_{r,s}^{\frac{1}{r-s}}$, as discussed in Lecture 2, satisfy the technical condition (3.22). We first treat the special case $f = H_r^{\frac{1}{r}}$ of higher order mean curvatures normalized to be of homogeneity one.

Lemma 3.7. *Let P be a homogeneous hyperbolic (with respect to $a = (1, \ldots, 1\ )$) of degree m with positive coefficients, normalized by $P(a) = 1$. Then $f = P^{\frac{1}{m}}$ satisfies (3.22).*

Proof. Using that f is homogeneous of degree 1 and concave,

$$f(\mu) \leq f(\kappa) + \sum (\mu_i - \kappa_i) f_i(\kappa) = \sum \mu_i f_i(\kappa)\ .$$

Choosing $\mu = \kappa^2 = ({\kappa_1}^2, \ldots, {\kappa_n}^2) \in \Gamma$ proves that $\sum \kappa_i^2\ f_i(\kappa) \geq f(\kappa^2)$. It remains to prove the inequality

$$f(\kappa^2) \geq f(\kappa)^2 \tag{3.23}$$

Applying Schwarz's inequality term by term to P gives

$$(P(\kappa))^2 \leq P(\kappa^2) P(a) = P(\kappa^2)\ .$$

Taking mth roots gives (3.22).

The argument for the curvature quotients uses some results from Lecture 2. We first show that it suffices to show that $H_{r,r-1}^{\frac{1}{r-1}}$ satisfies (3.22).

Lemma 3.8. *Let $f_1(\kappa), \ldots, f_N(\kappa)$ be admissible curvature functions all satisfying (3.22). Then $f = (\Pi_{k=1}^N f_k)^{\frac{1}{N}}$ is admissible and also satisfies (3.22).*

Proof. We have already shown in Lemma 2.12 that f is concave so we need only demonstrate (3.22). By a routine calculation and the arithmetic geometric mean inequality

$$\sum_{i=1}^n {\kappa_i}^2 \partial_i f = \frac{f}{N} \sum_i \sum_k \frac{{\kappa_i}^2 \partial_i f_k}{f_k} \geq \frac{f}{N} \sum_k f_k \geq f (\Pi f_k)^{\frac{1}{N}} = f^2$$

.

Proposition 3.9. $H_{r,r-1}^{\frac{1}{r-1}}$ *satisfies (3.22).*

Proof. using formula (2.5) of Lemma 2.14 we find

$$\sum {\kappa_i}^2 (S_{r-1} \partial_i S_r - S_r \partial_i S_{r-1}) = r S_r^2 - (r+1) S_{r-1} S_{r+1}\ . \tag{3.24}$$

Using that

$$S_r = \binom{n}{r} H_r \tag{3.25}$$

the Newton inequalities Proposition 2.7 may be rewritten as

$$(r+1)S_{r-1}S_{r+1} \le r(\frac{n-r}{n-r+1})S_r^2 \ . \tag{3.26}$$

Using (3.26) in (3.25) we find

$$\sum {\kappa_i}^2 \partial_i(\frac{S_r}{S_{r-1}}) \ge \frac{r}{n-r+1}(\frac{S_r}{S_{r-1}})^2 \ . \tag{3.27}$$

Again using (3.25) we see that (3.27) is equivalent to

$$\sum {\kappa_i}^2 \partial_i H_{r,r-1} \ge H_{r,r-1}^2 \tag{3.28}$$

and the proof is complete.

An important step in the method of moving spheres is the following maximum principle.

Proposition 3.10. *Let M be an elliptic Weingarten hypersurface with position vector X and unit normal ν oriented so that $f(\kappa) = c > 0$ and set $h = (|X|^2 + \frac{2}{c}X \cdot \nu)$. Then*

$$Lh = 2(\sum f_i - 1) - \frac{2}{c}(\sum \kappa_i^2 f_i - c^2)X \cdot \nu \tag{3.29}$$

In particular, $Lh \ge 0$ in $\{X \cdot \nu \le 0\}$.

Proof. By Theorem 3.4,

$$L(|X|^2) = 2c\ X \cdot \nu + 2\sum f_i \ ,$$

$$L(X \cdot \nu) = -c - \sum {\kappa_i}^2 f_i \ X \cdot \nu \ .$$

Multiplying the second equation by $\frac{2}{c}$ and adding gives (3.29). □

To fix the ideas, we will sketch a proof, using spherical reflection, of Alexandrov's theorem that an embedded closed Weingarten hypersurface in R^{n+1} is a sphere. For complete details and other applications, see [10]. Let M be a closed embedded Weingarten hypersurface with position vector X and let S_ρ of radius ρ and center at the origin (assume the origin lies in the unbounded component of the complement of S). For ρ large, $M \subset S_\rho$ and there is a first value ρ_0 where M is tangent to S_ρ. We decrease ρ and cut off a cap M_ρ and set $\tilde{M}_\rho = I(M_\rho)$.

Lemma 3.11. *([7]) The directions of principal curvatures of M_ρ map into the directions of principal curvature for $\tilde{M}_\rho$ and if $\tilde{\kappa}$ denotes the corresponding set of principal curvatures to κ,*

$$\tilde{\kappa} = \frac{1}{\rho^2}(|X|^2\kappa + 2X \cdot \nu \ \vec{1}) \ . \tag{3.30}$$

For $\rho < \rho_0$ but close to ρ_0 evidently $X \cdot \nu < 0$ and $\tilde{M}_\rho$ is contained inside M. We then let ρ_1 be the infimum of the values of ρ such that this property of $\tilde{M}_\rho$ holds. Just as in the standard case of Alexandrov reflection, we have to consider the possibilites that $\tilde{M}_{\rho_1}$ is tangent to M at an interior point (where $|X| > \rho_1$ and $X \cdot \nu < 0$) or at a boundary point (where $|X| = \rho_1$ and $X \cdot \nu = 0$). In either case, we want to show that $M \cap \{|X| < \rho_1\} = \tilde{M}_{\rho_1}$.

The following lemma will enable us to compare $f(\tilde{\kappa})$ and $f(\kappa)$ and prove a maximum principle.

Lemma 3.12. *Suppose $\kappa \in \Gamma$, $\tilde{\kappa} \in \Gamma$. Then*

$$f(\tilde{\kappa}) \leq \frac{|X|^2}{\rho^2} f(\kappa) + \frac{2X \cdot \nu}{\rho^2} \ . \tag{3.31}$$

Proof. By concavity and homogeneity of f,

$$\frac{|X|^2}{2\rho^2} f(\kappa) = f(\frac{1}{2}\tilde{\kappa} + \frac{1}{2}(\frac{-2X \cdot \nu}{\rho^2})\vec{1}) \geq \frac{1}{2} f(\tilde{\kappa}) + \frac{1}{2}(\frac{-2X \cdot \nu}{\rho^2})$$

which is equivalent to (3.31). □

Corollary 3.13. *Let M satisfy $f(\kappa) = c$.*
i. Suppose M^- is a component of $M \cap \{|X| > \rho\}$ on which $X \cdot \nu < 0$ and set $\tilde{M} = I(M^-)$. Then $f(\tilde{\kappa}$ lies outside $T = \{\lambda \in \Gamma : f(\lambda) \geq c\}$ unless M is a sphere of radius $\frac{1}{c}$.
ii. On $\partial\tilde{M}$, $f(\tilde{\kappa}) \leq c$.

Proof. From Lemma 3.12 and formula (3.31), if $\tilde{\kappa} \in T$, $f(\tilde{\kappa}) \leq \frac{c}{\rho^2}(|X|^2 + \frac{2}{c}X \cdot \nu) = \frac{c}{\rho^2}h$. By Proposition 3.10, h achieves its maximum on $S \cap \{|X| = \rho\}$ so $f(\tilde{\kappa}) \leq c$. If the inequality is strict on the interior of $\tilde{M}$, we have a contradiction. Otherwise, h has an interior maximum and so $h \equiv \rho^2$. This implies that all the principle curvatures of M^- have the value c and so M^- is a sphere of radius $\frac{1}{c}$. By unique continuation, this holds for M and so the first part of the lemma holds. The second part of the lemma is now clear by the preceding argument. □

Now suppose we are in the case of internal tangency. Then by a standard argument $\tilde{\kappa} \geq \kappa$ hence $f(\tilde{\kappa}) \geq c$. But $X \cdot \nu < 0$ so we are in the equality case of Corollary 3.13 so M is a sphere. In the case of boundary tangency, we have $|X| = \rho$ and $X \cdot \nu = 0$ at the point of tangency of M and $\tilde{M}_{\rho_1}$. In particular, $\tilde{\kappa} = \kappa$ by formula (3.30) so $\tilde{\kappa} \in \Gamma$ and so $f(\tilde{\kappa}) \leq c$ by Corollary(3.13) . We can write M and $\tilde{M}_{\rho_1}$ locally as graphs over their common tangent plane at the point of tangency with $\tilde{M}_{\rho_1}$ lying above M. Now we can apply the Hopf boundary point lemma to conclude that M is invariant under spherical reflection.

In all cases we have shown that for arbitrary center, M in invariant by reflection in some sphere with that center. By moving the center to infinity along a direction $\vec{n}$ we conclude in the limit that M is invariant by a hyperplane with normal $\vec{n}$ and so M is a sphere. For an interesting discussion of spherical symmetries see section 3 of [28].

Lecture 4. Monge-Ampere Boundary Value Problems and Applications

In this section we discuss boundary value problems in R^n and S^n for Monge-Ampère equations and discuss applications to existence questions for K-hypersurfaces.

Let $\Omega \subset R^n$ be a smooth domain and consider the Monge-Ampère equation

$$\begin{aligned} \det(u_{ij}) &= \psi(x,u,\nabla u) && \text{in } \Omega, \\ u &= \phi && \text{on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\psi > 0$ is smooth, $\phi \in C^\infty(\partial\Omega)$.

Remark 4.1. The choice $\psi = K(1+|\nabla u|^2)^{\frac{n+2}{2}}$ describes a graph $x_{n+1} = u(x)$ of constant Gauss curvature.

The classical PDE existence theorem is the following:

Theorem 4.2. *(*[3, 25]*) Suppose Ω is strictly convex and Ω , ψ , ϕ , smooth. Assume in addition that for the boundary data ϕ there is a strictly convex subsolution $\underline{u}$; i.e.*

$$\begin{aligned} \det(\underline{u}_{ij}) &\geq \psi(x,\underline{u},\nabla \underline{u}) && \text{in } \Omega, \\ \underline{u} &= \phi && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

Then there exists a strictly convex solution $u \in C^\infty(\overline{\Omega})$ a solution to (4.1). If $\psi_u \geq 0$ the solution is unique.

¿From the point of view of PDE, this result is essentially optimal but is it geometrically useful? The following examples show that the answer is not really.

Example 4.3. Let Γ_1 , Γ_0 be strictly convex smooth, closed codimension 2 hypersurfaces in parallel planes, say the planes $x_{n+1} = 1, 0$ respectively. We ask if there is a K-hypersurface solution for K sufficiently small? Intuitively, the answer is clearly yes. Let's specialize further and suppose that the parallel projection of Γ_1, call this projection γ_1, contains Γ_0. It is then not difficult to see that if a solution exists, it must be a graph over the annulus Ω with outer boundary γ_1 and inner boundary Γ_0 and so satisfies (4.1) with $\phi = 1$ on γ_1 and $\phi = 0$ on Γ_0. However since Ω is not convex, Theorem 4.2 does not apply! However, as shown in [22] there is a unique smooth solution as expected.

Example 4.4. Let S be an ovaloid in R^{n+1}, that is the boundary of a strictly convex body, and let D be a smooth domain on S with $\partial D = \Gamma = (\Gamma_1, \dots, \Gamma_m)$. We think of D as a strictly convex hypersurface with boundary Γ and ask if we can deform D to a K_0-hypersurface for $0 < K_0 \leq \inf_{P\in S} K(P)$? If yes, we expect that the solution should be a radial graph $X(x) = \rho(x)x$, x over a domain Ω contained in S^n obtained by projecting D radially (we choose an origin inside S) onto S^n. Moreover, $\rho(x) = \phi(x) > 0$ on $\partial\Omega$ where Γ is the radial graph of ϕ.

The Gauss curvature of X is given by (see [18])

$$K[X(x)] = \frac{\det(b_{ij})}{\det(g_{ij})} = \frac{\det(\rho^2\sigma_{ij} + 2\nabla_i\rho\nabla_j\rho - \rho\nabla_{ij}\rho)}{\sigma\,\rho^{2n-2}\,(\rho^2 + |\nabla\rho|^2)^{\frac{n+2}{2}}}. \tag{4.3}$$

where σ_{ij} is the standard metric on S^n and $\sigma = \det\sigma_{ij}$.

This expression for the Gauss curvature simplifies considerably if we instead consider the "dual" radial graph

$$u = \frac{1}{\rho}\ ,\ \varphi = \frac{1}{\phi}\ .$$

Then u is a solution to the Monge-Ampere type boundary value problem

$$\begin{array}{rcll} \sigma^{-1}\det(\nabla_{ij}u + u\sigma_{ij}) & = & K_0\frac{(u^2+|\nabla u|^2)^{\frac{n+2}{2}}}{u^{n+2}} & \text{in}\ \ \Omega, \\ u & = & \varphi & \text{on}\ \ \partial\Omega \end{array} \tag{4.4}$$

Again the point is that we cannot control the geometry of Ω so we must allow arbitrary geometry. Of course there are obstructions and we want to remove them by assuming the existence of an admissible subsolution. We will present two results from [18] for graphs over R^n and S^n respectively.

Theorem 4.5. *([18, 11]) Suppose Ω , ϕ , ψ are smooth and assume there is a locally strictly convex subsolution $\underline{u} \in C^\infty(\overline{\Omega})$, i.e,*

$$\begin{array}{rcll} \det\underline{u}_{ij} & \geq & \psi(x,\underline{u},\nabla\underline{u}) & \text{in}\ \Omega \\ \underline{u} & = & \phi & \text{on}\ \ \partial\Omega\ . \end{array} \tag{4.5}$$

Then there exists $u \in C^\infty(\overline{\Omega})$ a solution to (4.1). (If $\psi_u \geq 0$, there is uniqueness.) Moreover, any admissible solution satisfies the estimate $|u|_{C^{2+\alpha}(\Omega)} \leq C$ for a controlled constant C.

Theorem 4.6. *([18, 11]) Let $\Omega \subset S^n$ be a smooth domain that does not contain any hemisphere. Assume there is a locally strictly convex (i.e. $\nabla_{ij}\underline{u} + \underline{u}\ \sigma_{ij} > 0$) subsolution $\underline{u}$ to (4.4). Then $\exists u \in C^\infty(\overline{\Omega})$ a solution to (4.4). Moreover, any* *admissible solution satisfies the estimate* $|u|_{C^{2+\alpha}(\Omega)} \leq C$ *for a controlled constant* C.

Corollary 4.7. *Example 2 has a smooth solution inside* S *as conjectured.*

Proof. Choose an origin inside the convex hull of $S \setminus D$ and radially project D onto a subdomain Ω of S^n as described earlier. Then Ω does not contain any hemisphere and so Theorem 4.6 applies.

Corollary 4.8. *(polyhedral version of Example 2). Let* P *be a convex polyhedron in* R^{n+1} *and let* $\Gamma = (\Gamma_1, \dots, \Gamma_m)$ *be a collection of stricly convex closed hypersurfaces, each one contained interior to a face of* P *(with at most one in any face).Then for* $K_0 > 0$ *sufficiently small,* Γ *bounds a smooth embedded* K_0 *hypesurface which can be represented as a radial graph.*

Rob Kusner observed that Theorem 4.6 holds in greater generality and the proof is the same. Let U be a domain and let $x : U \to S^n$ be an immersion. Let ρ be a positive function on U and define $X(p) = \rho(p)x(p)$, $p \in U$. Consider the problem of finding a strictly locally convex immersed K-hypersurface:

$$X : U \to R^{n+1} \ , \ X(p) = \rho(p)x(p) \ , \ p \in U \ , \ \rho(p) = \phi(p) \ , \ p \in \partial U \ . \tag{4.6}$$

Theorem 4.9. *([18]) Suppose that no hemisphere can be isometrically embedded in* U *and . assume there is a smooth immersed locally strictly convex subsolution* $\overline{X} : U \to R^{n+1}$ *satisfying*

$$\overline{X}(p) = \overline{\rho}(p)x(p) \ , \ K(\overline{X}(p)) \geq K \ , \ p \in U \ , \ \overline{\rho} = \phi \text{ on } \partial U \ . \tag{4.7}$$

Then there exists a smooth immersed K-hypersurface X *satisfying (4.6) and* $\rho \leq \overline{\rho}$.

A more concrete form of Theorem 4.9 is given by

Corollary 4.10. *([18]) Let* $\Gamma = (\Gamma_1, \dots, \Gamma_m)$ *and suppose there exists an immersed strictly locally convex hypersurface* $\overline{X}$ *satisfying* $\overline{X} \cdot \nu < 0$ *(for suitable choice of origin) everywhere and such that no subdomain of* $\overline{X}$ *is radially projected injectively onto a hemisphere of* S^n *. Then for* K_0 *small enough there is an immersed* K_0*-immersed hypersurface spanning* Γ

Using Corollary 4.10 we can construct K-hypersurfaces of higher genus following a constructed suggested in [22].

Corollary 4.11. *([18]) For each positive integer k, there exists an embedded K-hypersurface of genus k.*

Before giving some idea of the proof of Theorem 4.5 (the proof of Theorem 4.6 is similar in spirit) we sketch its application [34] to the existence of complete embedded K-hypersurfaces in H^{n+1} with prescribed asymptotic boundary.

4.2. **Complete K-hypersurfaces in Hyperbolic space.** We use the half-space model

$$H^{n+1} = \{(x, x_{n+1}) = (x_1, \dots, x_{n+1}) : x_{n+1} \geq 0\} \text{ with metric } ds^2 = \frac{dx^2}{x_{n+1}^2} .$$

Suppose that we want to find a complete K-hypersurface with asymptotic boundary $\Gamma = \partial\Omega \subset \{x_{n+1} = 0\}$. In order to utilize Theorem 4.5 we vertically translate the domain Ω to the unit horosphere $P_1 = \{x_{n+1} = 1\}$ and look for a vertical graph $y = \log x_{n+1} = f(x)$, $x \in \Omega$. We then write down the first and second fundamental forms of the graph in order to express the extrinsic Gauss curvature K+1 as an expression in the second derivatives of f of Monge-Ampere type. The computation is straightforward but tedious and may be found in [34]. It turns out (just as in the proof of Theorem 4.6) that the equations simplify enormously if we use a new variable $u = e^{2f} = x_{n+1}^2$. We arrive at the following (degenerate) boundary value problem:

$$\det(u_{ij} + 2\delta_{ij}) = 2^n \, (K+1)(1 + \frac{|\nabla u|^2}{4u})^{\frac{n+2}{2}} \quad \text{in} \quad \Omega \tag{4.8}$$

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{4.9}$$

Although it is not essential, observe that equation (4.8) is the classical Mong-Ampere equations for the dependent variable $u + |x|^2$ and so Theorem 4.5 would apply except that the boundary $u = 0$ makes the right hand side possibly unbounded. Thus it is natural to approximate (4.8)(4.9) the desired K-hypersurface by a K-hypersurface with boundary Γ translated up to a horosphere $P_c = \{x_{n+1} = c\}$; that is, we want to solve

$$\det(u_{ij} + 2\delta_{ij}) = 2^n \, (K+1)(1 + \frac{|\nabla u|^2}{4u})^{\frac{n+2}{2}} \quad \text{in} \quad \Omega \tag{4.10}$$

$$u = c^2 \quad \text{on} \quad \partial\Omega \tag{4.11}$$

In order to apply Theorem 4.5, we need to find a subsolution to (4.12)(4.13). Since the horospheres are flat, $\underline{u} \equiv c^2$ is a strictly convex subsolution for $K \in (-1,0)$ (since $K+1>0$.) Thus we have shown

Proposition 4.12. *For $K \in (-1,0)$ there exists a smooth admissible solution $u_c \in C^\infty(\overline{\Omega})$ of (4.12)(4.13).*

One must now analyze the behavior of the family u_c as c tends to zero. The first step is to show $\frac{|\nabla u|^2}{4u}$ is uniformly bounded independent of c. The final result is

Theorem 4.13. *([34]) For $K \in (-1,0)$ there is a smooth admissible solution $u \in C^\infty(\Omega) \cap C^{1,1}(\bar{(\Omega)}$ to (4.8)(4.9). Hence there is a complete K-hypersurface S with asymptotic boundary $\Gamma = \partial\Omega$ which is a graph $y = \ln x_{n+1} = \frac{1}{2}\ln u$. Moreover, if $n=3$ and Γ is a Jordan curve, then S is unique among all possible immersed locally convex solutions.*

The uniqueness statement for $n=3$ arises because of the curious fact that the Jacobi operator $\tilde{L} = L + HK$ just as in R^{n+1} (see Theorem 3.4) which is elliptic since $H>0$ and $K<0$. This allows us to deform Γ from a small circle inside to a large circle outside and obtain a smooth family of solutions. The solution for a circle is unique (an equidistant sphere) and our family foliates the region of H^3 between the large sphere and the small sphere. Now suppose there was another K-hypersurface M with asymptotic boundary Γ. Because Γ is Jordan, it is not difficult to see that in fact M is embedded. Therefore the small equidistant sphere may be chosen inside M and the large equidistant sphere may be chosen to lie outside M. A simple maximum principle argument shows that S must be both inside M and outside M and so must equal M.

Theorem 4.13 has a more compelling interpretation in the ball model of H^3. Take Γ on the sphere at infinity and let C be the hyperbolic convex hull of Γ. Then for $K \in (-1,0)$ there is a unique embedded K-surface with asymptotic boundary Γ in each component of the complement of C. As K varies between zero and negative one, these solutions foliate each component going from the sphere to ∂C. Since ∂C is a hyperbolic surface in the sense of Thurston, our solutions provide smooth approximations with constant negative curvature.

4.3. **Apriori Estimates.** We now sketch the main ideas in the proof of Theorem 4.5. It is convenient to rewrite our equation (4.1) as

$$\log\det u_{ij} = \log\psi(x,u,\nabla u) := f(x,u,\nabla u) \tag{4.12}$$

noting that the operator on the left hand side of (4.12) is concave. We also define

$$L = u^{ij}\partial_i\partial_j - f_{p_i}\partial_i \tag{4.13}$$

which is the essential part of the full linearization of (4.12); here (u^{ij}) is the inverse matrix to the strictly positive matrix (u_{ij}).

Recalling that we assume the existence of a locally strictly convex subsolution $\underline{u}$ satisfying (4.2), we make the definition

Definition 4.14. The admissible class

$$A = \{w \in C^\infty(\overline{\Omega}) : (w_{ij}) > 0\ ,\ w \geq \underline{u}\ ,\ w = \underline{u} = \phi \quad \text{on} \quad \partial\Omega\}$$

Hence by the maxumum principle, $\underline{u} \leq w \leq h$ in $\overline{\Omega}$, where h is the harmonic extension of ϕ. It follows that we have the C^1 estimate

$$|w| + |\nabla w| \leq C \quad \text{in} \quad \overline{\Omega} \tag{4.14}$$

for a controlled constant C.

Theorem 4.15. *Let $u \in A$ be a solution to (4.1). Then $|D^2u| \leq C \quad in \quad \overline{\Omega}$ for a controlled constant C.*

Remark 4.16. Using the Evans-Krylov interior regularity theory (see [14]) and the boundary regularity results of [3], [4], [25], one can deduce from Theorem 4.15 a $C^{2+\alpha}(\overline{\Omega})$ estimate for u.

The essential step in the proof of Theorem 4.15 is the estimate on $\partial\Omega$:

Proposition 4.17. *Let $u \in A$ be a solution to (4.1). Then $|D^2u| \leq C \quad on \quad \partial\Omega$ for a controlled constant C.*

Proof. Step 1. Choose the origin of coordinates to be a point on $\partial\Omega$ (at which we will derive our estimates) with the x_n axis the interior normal direction and $x' = (x_1, \dots, x_{n-1})$ tangential. We locally write $\partial\Omega$ as a graph

$$x_n = \rho(x') = \frac{1}{2}\sum_{\alpha,\beta<n} B_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3)\ . \tag{4.15}$$

Since $u - \underline{u}(x', \rho(x')) \equiv 0$,

$$(u - \underline{u})_{\alpha\beta}(0) = -(u - \underline{u})_n(0)B_{\alpha\beta} \ ;$$

hence $|u_{\alpha\beta}(0)| \le C$
Step 2. We next show $|u_{\alpha n}(0)| \le C$. To this end, introduce the approximate tangential derivative

$$T_\alpha = \partial_\alpha + \sum_{\beta < n} B_{\alpha\beta}(x_\beta \partial_n - x_n \partial_\beta) \ ,$$

which differs from the exact tangential derivative $\partial_\alpha + \rho_\alpha \partial_n$ by $O(|x|^2)$. It is convenient to use $T = T_\alpha$ because it satisfies

$$L(Tu) = O(1) \tag{4.16}$$

as can be easily checked using the invariance of the left hand side of (4.12) under rotations of coordinates. From (4.16) it follows

$$|L(T(u - \underline{u}))| \le C(1 + \sum u^{ii}) \tag{4.17}$$

$$|T(u - \underline{u})| = O(|x|^2) \tag{4.18}$$

The following lemma from [16] constructs a comparison function v in terms of u , $\underline{u}$, h and the distance $d(x, \partial\Omega)$ in a small neighborhood $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$ of $\partial\Omega$. The construction is much simplified if we were to assume that $\det \underline{u}_{ij} \ge \psi(x, \underline{u}, \nabla \underline{u}) + \epsilon$ as in [18].

Lemma 4.18. *Let $v = (u - \underline{u}) + t(h - \underline{u}) - Nd(x, \partial\Omega)^2$ in Ω_{delta}. Then for t , δ sufficiently small and N sufficiently large,*

$$Lv \ \le \ -\frac{\epsilon}{4}(1 + \sum u^{ii}) \tag{4.19}$$

$$v \ \ge \ 0 \quad on \quad \partial\Omega_\delta \tag{4.20}$$

$$v \ = \ 0 \quad on \quad \partial\Omega \tag{4.21}$$

Now we can choose $A >> B >> 1$ such that

$$L(Av + B|x|^2 \pm T(u - \underline{u})) \le 0$$

and

$$Av + B|x|^2 \pm T(u - \underline{u}) \ge \quad \text{on} \quad \partial\Omega_\delta \ .$$

It follows from the maximum principle that

$$|u_{\alpha n}(0)| \leq C$$

Step 3. It remains to estimate $u_{nn}(0)$. Here we use the equation

$$A^{ij}u_{ij} = n\det u_{ij} = O(1) \ ,$$

(where A^{ij} is the cofactor matrix of (u_{ij})) to solve for u_{nn} in terms of the derivatives we have already estimated. We can do this if we know that A^{nn} is strictly positive, i.e. we need to know the strict tangential convexity of u. This strict tangential convexity in fact holds for all $w \in A$:

Proposition 4.19. *([18])*$\sum_{\alpha,\beta<n}(0)\xi_\alpha\xi_\beta \geq c_0 > 0$ $\forall w \in A$ *for a controlled constant* c_0.

Of course, the point here is that no assumption is made on the geometry of $\partial\Omega$. For a proof we refer the reader to [18]

Remark 4.20. The case $\partial\Omega$ concave is actually the easiest since

$$(u - \underline{u})_{\alpha\alpha} = -(u - \underline{u})_n B_{\alpha\alpha} \geq 0$$

since $(u - \underline{u})_n > 0$ and $B_{\alpha\alpha} \leq 0$.

This completes the sketch of the proof of Proposition 4.17. □

4.4. **Degenerate Monge-Ampere equations and convex hulls.** In order to get a better understanding of the general existence problem for K-hypersurfaces posed at the beginning of this article, it is of great interest to study totally degenerate Monge–Ampère equations. For a single smooth closed codimension 2 embedded submanifold Γ of R^{n+1}, this corresponds to the geometric question of the existence of convex hypersurfaces $S^\pm$ with Gauss curvature $K(S^\pm) \equiv 0$. The hypersurfaces $S^\pm$ correspond to the boundaries of the convex hull $C(\Gamma)$ which is the convex region bounded by the two "convex caps" $S^\pm$ which meet along Γ. It is well-known that $S^\pm$ cannot be smooth, in general.

For Γ a graph over the boundary of a strictly convex domain Ω the lower cap S can be represented as a graph $x_{n+1} = u(x)$ where

$$u(x) = \max\{v(x)|v(x) \in C^\circ(\bar{\Omega}), v \text{ convex}, \ v \leq \phi \text{ on } \partial\Omega\}$$

(here Γ = graph ϕ over $\partial\Omega$).

The convex function u is a weak solution (in the Alexandrov sense or the viscosity sense) of the degenerate Monge–Ampère boundary value problem:

$$\begin{aligned} \det u_{ij} &\equiv 0 \quad \text{in } \Omega \\ u &= \phi \quad \text{on } \partial\Omega \,. \end{aligned} \tag{4.22}$$

How regular is u? Many people have studied this question including Rauch–Taylor [30], Trudinger–Urbas [37] and Caffarelli–Nirenberg–Spruck [4]. The optimal regularity was obtained in [8].

Theorem 4.21 ([8])**.** *Assume Ω strictly convex with $\partial\Omega \in C^{3,1}$ and let $\phi \in C^{3,1}$. Then the unique admissible weak solution u of* (4.22) *is in $C^{1,1}(\bar{\Omega})$.*

The simple example $u = (1+y)^{2-\varepsilon}$, $\varepsilon \in (0,1)$ on $B_1(0) \subset \mathbf{R}^2$ taken from [8] shows that this result is optimal.

An important generalization of this result was obtained by Guan [16]. Here, as in [18] we drop the assumption of strict convexity of Ω but assume that there is an admissible subsolution $\underline{u}$ in Ω for the given boundary data ϕ. More precisely,

Theorem 4.22 ([16])**.** *Let Ω be a $C^{3,1}$ domain and $\phi \in C^{3,1}(\partial\Omega)$. Suppose $\exists$ a locally strictly convex function $\underline{u} \in C^2(\bar{\Omega})$ with $u = \phi$ on $\partial\Omega$. Then $\exists!$ locally convex weak solution of* (4.22) *in $C^{1,1}(\bar{\Omega})$.*

The extension of Guan's result to space curves (or codimension 2 submanifolds) was obtained by Ghomi [12][13].

Definition 4.23. Suppose Γ is a smooth Jordan curve lying on an ovaloid O. Then

i) Γ is strictly convex, that is, through every point $x_0 \in \Gamma$, there exists a supporting plane H_{x_0} with $H_{x_0} \cap \Gamma = \{x_0\}$ and

ii) the curvature $k(\Gamma) \neq 0$, i.e., Γ has no inflection points.

We will call a Jordan curve Γ satisfying i,ii strictly convex. More generally, an m-dimensional closed embedded submanifold Γ of R^{n+1} is called strictly convex if

i′) each point $x_0 \in \Gamma$ has a strict support plane H_{x_0,n_0} with $\Gamma \backslash \{x_0\}$ contained in one of the open half-spaces determined by H, say $\ell(x) = \langle x - x_0, n_0 \rangle < 0$ $\forall x \neq x_0$, $x \in \Gamma$ (n_0 is the outer normal) and

ii′) H_{x_0,n_0} is non-singular, that is, x_0 is a non-degenerate critical point of ℓ. This last condition is equivalent to the condition that $\langle A_{n_0} X, X \rangle < 0 \;\; \forall X \in T_{x_0}\Gamma$, $X \neq 0$, where A_{n_0} is the second fundamental form of Γ at x_0 with respect the normal direction n_0.

In his thesis, Ghomi proves the following

Theorem 4.24 ([12, 13]). *Every $C^{k,\alpha}$ strictly convex submanifold Γ lies on a $C^{k,\alpha}$ strictly convex ovaloid O, $\alpha \in [0,1]$.*

Now consider Γ, a strictly convex codimension 2 in the sense of Ghomi, with $\Gamma \in C^{3,1}$. Then Γ lies on a $C^{2,1}$ ovaloid O and we let $D^{\pm}$ denote the two components of the complement of Γ on O. Choose one of these components, say D^- and choose an origin inside the convex hull of $O - D^-$ so that D^- radially projects onto a domain Ω contained in a hemisphere (say the upper hemisphere) of the unit sphere $S^n \subset R^{n+1}$.

We look for S^-, the component of $\partial C(\Gamma)$ "facing D^-" as a radial graph over Ω :

$$\begin{aligned} X &= \rho(x)x\ , \quad x \in \Omega \\ \rho &= \phi \qquad\quad\ \text{on } \partial\Omega \end{aligned}$$

where $\Gamma = \phi(x)x$, $x \in \partial\Omega$. By assumption, $\Omega, \phi \in C^{3,1}$. As explained earlier, it is simpler to work with $u = \frac{1}{\rho}$; then the Gauss curvature $K(X)$ is related to u by

$$\begin{aligned} \frac{\det(\nabla_{ij}u + u\sigma_{ij})}{\sigma} &= \frac{K(u^2 + |\nabla u|^2)^{\frac{n+2}{2}}}{u^{n+2}} \quad \text{in} \quad \Omega \\ u = \varphi &= \frac{1}{\phi} \qquad\qquad\qquad \text{on} \quad \partial\Omega\ . \end{aligned} \tag{4.23}$$

Here σ_{ij} denotes the metric on S^n, $\sigma = \det \sigma_{ij}$ and $\nabla_{ij}u$ is the Hessian of u with respect to the metric σ_{ij}.

Thus we are looking for an admissible ($\nabla_{ij}u + u\sigma_{ij} \geq 0$) weak solution of

$$\begin{aligned} \det(\nabla_{ij}u + u\sigma_{ij}) \equiv 0 \quad &\text{in} \quad \Omega \\ u = \varphi \qquad\qquad &\text{on} \quad \partial\Omega \end{aligned} \tag{4.24}$$

(where $\partial\Omega, \varphi \in C^{3,1}$).

It is convenient to approximate (4.24) with the nondegenerate problems

$$\begin{aligned} \frac{\det(\nabla_{ij}u + u\sigma_{ij})}{\sigma} = \varepsilon\mu^{n+2} \quad &\text{in} \quad \Omega \\ u = \varphi \qquad\qquad &\text{on} \quad \partial\Omega \end{aligned} \tag{4.25}$$

where μ is a fixed positive smooth function we will define in a moment. Since D^- is strictly convex there exists a strictly convex subsolution $u \in C^{3,1}(\bar{\Omega})$, $u = \varphi$ on $\partial\Omega$ for $0 < \varepsilon \ll 1$. Applying the results of [18] there exists $u^\varepsilon \in C^{3,\alpha}(\bar{\Omega})$ an admissible solution of (4.25) (we will see presently that u^ε is unique) and $|\nabla u^\varepsilon| \leq C$ independent of ε. Therefore u^ε converges uniformly to u an admissible weak solution of (4.24) (also unique).

Theorem 4.25. $u \in C^{1,1}(\bar{\Omega})$

It seems difficult (but probably possible) to extend Guan's result to domains Ω of arbitrary geometry in S^n. However, we can proceed as follows. We parametrize the upper hemisphere S^n_+ by choosing x to lie in the tangent plane to S^n at the north pole and setting

$$y = \frac{(x,1)}{\mu(x)} \in S^n,\ \mu(x) = \sqrt{1+x^2}\ .$$

Then

$$\sigma_{ij} = \langle y_{x_i}, y_{x_j} \rangle = \tfrac{1}{\mu^2}(\delta_{ij} - \tfrac{x_i x_j}{\mu_2})$$
$$\sigma^{ij} = (\sigma_{ij})^{-1} = \mu^2(\delta_{ij} + x_i x_j)$$
$$\sigma = \det \sigma_{ij} = \mu^{-(2n+2)}$$
$$\Gamma^k_{ij} = -\tfrac{1}{\mu^2}(x_i \delta_{kj} + x_j \delta_{ki})\ .$$

Set $\tilde{u}^\varepsilon(x) = \mu(x) u^\varepsilon(y(x))$ (the degree 1 homogeneous extension of u restricted to the tangent plane). Then a calculation gives

$$\tilde{u}^\varepsilon_{x_i x_j} = \mu(\nabla_{ij} u^\varepsilon + u^\varepsilon \sigma_{ij})\ ,$$

and so

$$\begin{aligned} \det \tilde{u}^\varepsilon_{x_i x_j} &= \sigma \mu^n \tfrac{\det(\nabla_{ij} u^\varepsilon + u^\varepsilon \sigma_{ij})}{\sigma} \\ &= \mu^{-(n+2)} \varepsilon \mu^{n+2} \equiv \varepsilon \text{ in } \tilde{\Omega} \\ \tilde{u}^\varepsilon &= \tilde{\varphi} \equiv \mu\varphi \text{ on } \partial\tilde{\Omega} \end{aligned} \tag{4.26}$$

where $\tilde{\Omega}$ is the central projection of Ω from the origin onto the tangent plane at the north pole. Moreover, there exists a strictly convex subsolution $v = \tilde{U} = \mu U$ in $\tilde{\Omega}$ with the given boundary data $v = \tilde{\varphi} \equiv \mu\varphi$ on $\partial\tilde{\Omega}$ and the $\tilde{u}^\varepsilon$ converge uniformly to $\tilde{u} = \mu u$ a convex weak solution of

$$\begin{aligned} \det \tilde{u}_{ij} &\equiv 0 \quad \text{in } \tilde{\Omega} \\ \tilde{u} &= \tilde{\varphi} \quad \text{on } \partial\tilde{\Omega} \end{aligned} \tag{4.27}$$

where $\tilde{\Omega}$, $\tilde{\varphi} \in C^{3,1}$.

Applying Guan's results [16, Theorem 1.1] gives $\tilde{u} \in C^{1,1}(\bar{\tilde{\Omega}})$. Thus $u \in C^{1,1}(\bar{\Omega})$ as required.

Corollary 4.26. *S^- is $C^{1,1}$ up to Γ .*

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