TWO DIMENSIONAL MINIMAL GRAPHS OVER UNBOUNDED DOMAINS

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Abstract. In this paper we will study solution pairs (u, D) of the minimal surface equation defined over an unbounded domain D in R^2, with $u = 0$ on ∂D. It is well known that there are severe limitations on the geometry of D; for example D cannot be contained in any proper wedge (angle less than π). Under the assumption of sublinear growth in a suitably strong sense, we show that if u has order of growth α in the sense of complex variables, then the "asymptototic angle" of D must be at least $\frac{\pi}{\alpha}$. In particular, there are at most two such solution pairs defined over disjoint domains. If $\alpha < 1$ then u cannot change sign and there is no other disjoint solution pair. This result is sharp as can be seen by a suitable piece of Enneper's surface which has order $\alpha = \frac{2}{3}$ and asymptotic angle $\frac{3\pi}{2}$.

1. Introduction

In this paper we consider solutions of the minimal surface equation

$$\sum_{i,j=1}^{2}(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2})u_{ij} = 0 \text{ in } D \tag{1.1}$$

$$u = 0 \text{ on } \partial D \ . \tag{1.2}$$

where D is an unbounded domain in R^2. Theorems limiting the behavior of solution pairs are of great utility in the study of complete embedded minimal surfaces in R^3 (for example, [4],[5]). We will see that there are severe limitations on the possible solution pairs (u, D) , $u \not\equiv 0$.

For example, if D is contained in a proper wedge (angle less than π), then no nontrivial solution pair exists [6]. The idea of the proof is to compare u with a rescaled Scherk graph. More precisely, let the vertex of the wedge be the origin and

1991 *Mathematics Subject Classification.* Primary 35J60, 53A10.

Key words and phrases. Minimal Surfaces, Differential geometry .

The author would like to thank Université Paris VII and The Mathematics Institute of Jussieu for its hospitality during the 1999-2000 academic year. Research partially supported by NSF Grant DMS-0072242 .

let T_1 be the isoceles triangle obtained by joining the points on the boundary of the wedge at distance 1 from the origin. Over T_1 there is a "Scherk" solution v of the minimal surface equation with boundary values 0 on the two sides of length 1 and $+\infty$ on the third side; this solution exists by the work of Jenkins- Serrin [9]. Let $w = Rv(\frac{x}{R})$ be the rescaled solution defined in $T_R = RT_1$. By the maximum principle, $|u| \leq w$ in $T_R \cap D$. Near the origin, $v = O(|x|^{\frac{\pi}{\gamma}})$, where $\gamma < \pi$ is the wedge angle (that is, w behaves like a harmonic function). Hence $w \leq CR^{(1-\frac{\pi}{\gamma})}$ on compact subsets of D. By letting R tend to ∞, we see that $u \equiv 0$ in D.

This leads one to suspect that in some measure theoretic sense, D must open up to an asymptotic angle of at least π in order to support a solution u vanishing on ∂D. This should imply that there are at most two nontrivial solution pairs over disjoint domains (one in the case of sublinear growth) and has been conjectured by Meeks.

In this note, we will prove Meek's conjecture under additional assumptions.

Definition 1.1. We say that a solution pair (u, D) of (1.1),(1.2) is an admissible solution if (i) $|\nabla u(z)| \to 0$
(ii) $|K(z, u(z))| \leq \frac{C}{1+|z|^2}$
as $z \in D$ tends to ∞, where K is the Gauss curvature of the graph.
(iii) $\partial D \cap \{|z| = \rho\} \neq \phi$ for all ρ sufficiently large.

Remark 1.2. 1. The condition on the Gauss curvature of the graph is the natural flatness condition for invariance under scaling.
2. If $W = \sqrt{1 + |\nabla u|^2}$, then (i)and (ii) imply $|\nabla W| = o(\frac{1}{|z|})$ as $z \in D$ tends to ∞.
3. Condition (iii) rules out the uninteresting case of the exterior of a finite number of disjoint compact domains. It is also needed in the use of Wirtinger's inequality in Lemma 2.1

Definition 1.3. 1. Let (u, D) be an admissible solution pair. The order α of u in D is given by

$$\alpha = \limsup_{z \in D\ ,\ z \to \infty} \frac{\log |u(z)|}{\log |z|}\ .$$

2. The asymptotic angle β of D is defined by

$$\beta = \limsup_{\rho\to\infty} \frac{1}{\rho}|D \cap \{|z = \rho|\}|$$

Remark 1.4. Note that by our definition of admissibility, $\alpha \leq 1$. Note also that if D' is a component of $\{z \in D : u(z) \neq 0\}$, then (u, D') is an admissible solution pair.

Theorem 1.5. *Let (u, D) be an admissible solution pair with order $\alpha \leq 1$. Then the asymptotic angle $\beta \geq \frac{\pi}{\alpha}$. Hence if $\alpha < 1$, u cannot change sign and there is no other admissible solution pair (u', D') with D' disjoint from D. If $\alpha = 1$, either u is of one sign with at most one other disjoint solution pair (u', D') (necessarily u' of one sign and $\alpha' = 1$) or $\{z \in D : u(z) \neq 0\}$ has exactly two components and there are no other disjoint solution pairs.*

Remark 1.6. A version of Theorem 1.5 in R^n also follows by the same method using the Faber-Krahn inequality in place of the Wirtinger equality. In condition (ii) of admissibility $|K|$ is replaced by $|A|$, the norm of the second fundamental form of the graph of u.

Example 1.7. Enneper's minimal surface provides a very concrete illustration of Theorem 1.5. It is a complete properly immersed minimal surface given explicitly by the Weierstrass-Enneper representation:

$$X(w) = (x_1, x_2, x_3) = \Re \int_0^w [(1 - g^2), i(1 + g^2), 2g], f\ dw$$

with $f(w) \equiv 1$ and $g(w) = w$. Integration gives

$$z = \overline{w} - \frac{w^3}{3}$$
$$x_3 = \Re w^2$$

where $z = x_1 + ix_2$ and $w = u + iv$.

Now let $\Omega = \{(u, v) : u^2 - v^2 > 2\ , -u < v < u\ \}$, that is, Ω is a simply connected domain in the first and fourth quadrant where $z > 2$. It is not difficult to check that the map $z(w)$ restricted to Ω is a diffeomorphism and that the image domain D is asymtotically a wedge of angle $\beta = \frac{3\pi}{2}$. Therefore $\phi(x_1, x_2) = x_3(w(z)) - 2$ is a solution of the minimal surface equation in D with $\phi = 0$ on ∂D. In other words, the pair (ϕ, D) is admissible with order $\alpha = \frac{2}{3}$. Note that $\beta = \frac{3}{2}\pi = \frac{\pi}{\alpha}$.

Example 1.8. If we cut the catenoid by a plane through the axis of symmetry, we obtain a solution pair (u, D) with $u > 0$ of exponential growth in the domain $D = \{(x, y) : |y| < \cosh x\}$. Let m be the maximum of $u(0, y)$, $|y| \leq 1$ and let D' be the component of $\{(x, y) \in D : u(x, y) > m + 1\}$ contained in $\{x > 0\}$. Then $(u - m - 1\ ,\ D')$ is a solution pair with asymptotic angle π and this angle is approached exponentially fast. Michael Beeson asked if there is any solution pair contained in the domain $\Omega = \{(x, y) : |y| < Ax^{2N} + B\}$ for arbitrary positive N,A,B. By extending the argument for the proper wedge, we can see that the answer is no as follows. Without loss of generality we may assume A=B=1. Suppose (u,D) is a solution pair with D contained in Ω. Arguing as for the catenoid, we may assume that D is contained in $\Omega \cap \{x > 0\}$. Let T_R be the isoceles triangle with vertex $(-R^{\frac{1}{2N}-1}, 0)$ and symmetric vertices $(R^{\frac{1}{2N}}, R+1), (R^{\frac{1}{2N}}, -R-1)$. Note that the vertex angle γ is approximately $\pi - 2R^{\frac{1}{2N}-1}$. Arguing as before, we find that $0 < u(x) < Cx$, that is, u has at most linear growth. By the proof of the half-space theorem of Hoffman-Meeks [8] this is impossible (the graph of u cannot have a contact at a finite point or at infinity with the plane $z = Lx$).

It is useful to have a variant of Theorem 1.5 where we only insist that $u = 0$ on ∂D outside a compact set.

Theorem 1.9. *Suppose that the boundary condition $u = 0$ on ∂D holds outside a compact set. Then Theorem 1.5 remains valid if*

$$\limsup_{\rho \to \infty} \rho \int_{D \cap \{|z = \rho|\}} u^2 \; d\theta = +\infty$$

Remark 1.10. Theorem 1.9 is false without the growth condition on u. For in any strictly convex unbounded domain D, we can prescribe $u = \phi$ on ∂D with $\phi \geq 0$ everywhere and $\phi = 0$ outside the ball of radius 2. The existence of such a solution is proven in [6]. Evidently, u decays to zero at infinity.

2. Notation and Preliminaries

For $\rho \geq \rho_o$ sufficiently large, let $D_\rho = D \cap \{|z| < \rho\}$ and $C_\rho = D \cap \{|z| = \rho\} \neq \phi$. We write $C_\rho = \bigcup_{i=1}^{N_\rho} C_\rho^i$ as a finite union. We denote the linear measure

$$|C_\rho^i| = 2\pi\theta^i(\rho)\ ;\ 0 < \theta^i(\rho) \leq 1$$

and introduce $\theta(\rho), I(\rho)$ and $E(\rho)$:

$$\begin{aligned}\theta(\rho) &= \sum_i \theta^i(\rho)\\ I(\rho) &= \sum_i \int_{C_\rho^i} \frac{u^2}{W} d\theta\\ E(\rho) &= \int_{D_\rho} \frac{|\nabla u|^2}{W} dx = \int_{C_\rho} \rho \frac{uu_r}{W} d\theta\ .\end{aligned}$$

In the remainder of the paper we will write E, I, etc and not indicate the dependence on ρ.

Lemma 2.1.

$$\int_{C_\rho} \frac{u_\theta^2}{W} d\theta \geq \frac{1+o(1)}{4\theta^2}\ I\ . \tag{2.1}$$

$$\int_{C_\rho} \rho^2 \frac{u_r^2}{W} d\theta \geq \frac{E^2}{I}\ . \tag{2.2}$$

$$\rho E' \geq \frac{E^2}{I} + \frac{1+o(1)}{4\theta^2(\rho)}\ I \tag{2.3}$$

$$\rho I' = 2E(\rho) + o(1) I\ . \tag{2.4}$$

Proof. We use the classical Wirtinger inequality on C_ρ^i:

$$\int_{C_\rho^i} u_\theta^2\ d\theta \geq \frac{1}{4\theta^i(\rho)^2} \int_{C_\rho^i} u^2 d\theta$$

For ρ large, this implies

$$\int_{C_\rho} \frac{u_\theta^2}{W} d\theta \geq \frac{1+o(1)}{4\theta(\rho)^2}\ I$$

for (u, D) admissible, proving the first assertion. The estimate on E is standard:

$$E^2 \le \sum \int_{C^i_\rho} \rho^2 \frac{u_r^2}{W}\, d\theta \int_{C_{\rho^i}} \frac{u^2}{W} d\theta \le \int_{C_\rho} \frac{\rho^2 u_r^2}{W} d\theta \ I \ .$$

Dividing both sides by I proves the second assertion. For the third assertion,

$$\rho E' = \sum_i \int_{C^i_\rho} (\rho^2 \frac{u_r^2}{W} + \frac{u_\theta^2}{W})\, d\theta \ .$$

Hence using (2.1)(2.2),

$$\rho E' \ge \frac{E^2}{I} + \frac{1+o(1)}{4\theta^2(\rho)}\ I$$

Finally,

$$I' = \int_{C_\rho} \frac{2uu_r}{W} d\theta + \int_{C_\rho} u^2 \left(\frac{\partial}{\partial r} \frac{1}{W} \right) d\theta$$

Since $\frac{\partial}{\partial r}\frac{1}{W} = o(\frac{1}{\rho})$ by Remark 1.2, the last assertion follows.

3. The frequency function

The method of frequency functions has been extensively utilized in recent years [1], [7] [3], mostly for the study of local regularity or issues of unique continuation. Here we use it to derive a precise asymptotic relationship between $\theta(\rho)$ and the order α of u .

Definition 3.1. The "frequency function" is defined by $U = \frac{E}{I}$.

Lemma 3.2. $\rho U' + (1+o(1))U^2 \ge \frac{1+o(1)}{4\theta(\rho)^2}$.

Proof. Writing $E = UI$, we have from Lemma 2.1:

$$\rho E' = \rho(IU' + UI') \ge \frac{U^2 I^2}{I} + \frac{(1+o(1))}{4\theta(\rho)^2} I$$

or

$$\rho U' + U \frac{\rho I'}{I} \ge U^2 + \frac{1+o(1)}{4\theta(\rho)^2}.$$

Recalling $\rho \frac{I'}{I} = 2U + o(1)$, we obtain

$$\rho U' + (U + o(1))^2 \ge \frac{1+o(1)}{4\theta(\rho)^2} \ .$$

Finally, from $U^2 + o(1)U \le (1+o(1))U^2 + o(1)$ the lemma follows since the $o(1)$ term can be absorbed into the right hand side.

Lemma 3.3.
$$U \geq \frac{1}{2} + o(1)$$
as $\rho \to \infty$

Proof. Fix $0 < \varepsilon << 1$. Using Lemma 3.2 we consider two cases:

Case 1. $\rho_1 U'(\rho_1) \leq \varepsilon$ for some $\rho_1 \geq \rho_0$. Then $U(\rho_1) \geq \frac{1}{2} - \varepsilon + o(1)$. Hence U must stay above $\frac{1-\varepsilon}{2} + o(1)$ for $\rho > \rho_1$.

Case 2. $\rho U' > \varepsilon \quad$ for $\quad \rho \geq \rho_0$. Then
$$U \geq \varepsilon \log \frac{\rho}{\rho_o} + U(\rho_0) \to \infty \quad \text{as} \quad \rho \to \infty\ .$$
Since ε is arbitrary, the lemma is proven.

Corollary 3.4. $\rho \frac{I'}{I} \geq 1 + o(1)$ *as* $\rho \to \infty$. *Moreover, the order* α *of* u *is at least* $\frac{1}{2}$.

Proof. Since $\rho \frac{I'}{I} = 2U + o(1)$, the first part follows from Lemma 3.3. Fix $0 < \varepsilon << 1$ and ρ_o so large that
$$\frac{I'}{I} \geq \frac{1-\varepsilon}{\rho}\ ,\ \rho \geq \rho_o$$
Integration gives,
$$\log \frac{I}{I(\rho_o)} \geq (1-\varepsilon) \log \frac{\rho}{\rho_o}\ .$$
Let $M(\rho) = \sup_{C_\rho} |u|$ and observe $I \leq 2\pi M^2(\rho)$. Hence
$$2 \frac{\log M(\rho)}{\log \rho} \geq (1-\varepsilon) + O(\frac{1}{\log \rho})\ .$$
Letting $\rho \to \infty$ gives $\alpha_D(u) \geq \frac{1-\varepsilon}{2}$. Since ε is arbitrary, we find $\alpha \geq \frac{1}{2}$ as claimed.

Lemma 3.5.
$$\frac{\rho U'}{U} + 2U \geq \frac{1 - o(1)}{\theta(\rho)}$$

Proof. Using Lemma 3.2,
$$(1 + o(1)) \frac{\rho U'}{U} + ((1 + o(1))U)^2 \geq \frac{1 + o(1)}{4\theta(\rho)^2}$$
Hence,
$$(\frac{\rho U'}{2U} + (1 + o(1))U)^2 \geq \frac{\rho^2 U'^2}{4U^2} + \frac{1 + o(1)}{4\theta(\rho)^2}$$

Using the inequality

$$\sqrt{(a^2+b^2)} \geq \varepsilon a + \sqrt{(1-\varepsilon^2)}\ b$$

with $a = |\frac{\rho U'}{2U}|$, $b = \frac{\sqrt{1+o(1)}}{2\theta(\rho)}$ and $\varepsilon = o(1)$, we obtain

$$\frac{\rho U'}{2U} + (1+o(1))U \geq o(1)\ |\frac{\rho U'}{2U}| + \frac{1+o(1)}{2\theta(\rho)}$$

and this implies

$$(1+o(1))(\frac{\rho U'}{U} + 2U(\rho)) \geq \frac{1+o(1)}{2\theta(\rho)}\ .$$

Multiplying both sides by $\frac{2}{(1+o(1))}$ proves the lemma.

Corollary 3.6.

$$\rho\frac{E'}{E} = \frac{\rho U'}{U} + \frac{\rho I'}{I} \geq \frac{(1+o(1))}{\theta(\rho)}$$

Proof. Follows immediately from $\frac{\rho I'}{I} = 2U + o(1)$.

Proposition 3.7. *For* $\rho_o \leq \frac{\rho}{2}$,

$$I \geq I(\rho_0) + c_1 e^{\int_{\rho_0}^{\rho/2}(1+o(1))\frac{1}{\lambda\theta(\lambda)}d\lambda}$$

where $c_1 = 2\ln 2(1+o(1))E(\rho_o)$.

Proof. From Corollary 3.6,

$$2E \geq 2E(\rho_o)e^{\int_{\rho_o}^{\rho}(1+o(1))\frac{1}{\lambda\theta(\lambda)}\ d\lambda}$$

On the other hand, $\frac{\rho I'}{I} = 2U + o(1) \geq 1 + o(1)$ and so

$$\begin{aligned} I &\leq (1+o(1))\rho I' \\ \rho I' = 2E + o(1)I &\geq 2E + o(1)\rho I'\ . \end{aligned}$$

Hence,

$$\rho I' \geq c_o e^{\int_{\rho_o}^{\rho}(1+o(1))\frac{1}{\lambda\theta(\lambda)}\ d\lambda}\ ,\ c_o = 2(1+o(1))E(\rho_o)\ . \tag{3.1}$$

Taking $\frac{\rho}{2} \geq \rho_o$ and integrating (3.1) gives

$$\begin{aligned} I &\geq I(\rho_o) + c_o \int_{\rho_o}^{\rho} \frac{1}{t} e^{\int_{\rho_o}^{t} \frac{1+o(1)}{\lambda\theta(\lambda)}\ d\lambda}\ dt \\ &\geq I(\rho_o) + c_o \int_{\frac{\rho}{2}}^{\rho} \frac{1}{t} e^{\int_{\rho_o}^{t} \frac{1+o(1)}{\lambda\theta(\lambda)}\ d\lambda}\ dt \\ &\geq I(\rho_o) + c_o\ \ln 2\ e^{\int_{\rho_o}^{\frac{\rho}{2}} \frac{1+o(1)}{\lambda\theta(\lambda)}\ d\lambda} \end{aligned}$$

4. Proof of the Main Theorem

The main theorem of the paper will follow from

Theorem 4.1. *Let (u, D) be an admissible solution pair and let $\varepsilon > 0$ be fixed. Then for ρ_o sufficiently large,*

$$\frac{1}{\alpha+\varepsilon} \leq \frac{2\log\rho}{\left(\log\frac{\rho}{2\rho_0}\right)^2} \int_{\rho_0}^{\rho/2} (1+o(1))\ \frac{\theta(\lambda)}{\lambda} d\lambda \leq (1+o(1))\frac{\beta}{\pi}\ . \tag{4.1}$$

In particular, $\beta \geq \frac{\pi}{\alpha}$.

Proof. Fix $0 < \varepsilon << 1$ and let α be the order of u in D. Then $2\log M(\rho) \leq 2(\alpha + \varepsilon)\log\rho$ for $\rho = |z|$ large enough. On the other hand, $I(\rho) \leq 2\pi\theta(\rho)M(\rho)^2 \leq 2\pi M^2$. Hence using Proposition 3.7,

$$2(\alpha+\varepsilon)\log\rho \geq 2\log M \geq \int_{\rho_0}^{\rho/2} \frac{(1+o(1))}{\lambda\theta(\lambda)}\ d\lambda\ . \tag{4.2}$$

To proceed further, we rewrite (4.2) as

$$\frac{1}{\alpha+\varepsilon} \leq \frac{2\log\rho}{\int_{\rho_0}^{\rho/2}(1+o(1))\frac{d\lambda}{\lambda\theta(\rho)}} \tag{4.3}$$

Now by Schwartz's inequality,

$$(\log\frac{\rho}{2\rho_0})^2 \leq \int_{\rho_0}^{\rho/2} (1+o(1))\ \frac{\theta(\lambda)}{\lambda} d\lambda \int_{\rho_0}^{\rho/2} (1+o(1))\ \frac{d\lambda}{\lambda\theta(\lambda)}$$

or

$$\frac{1}{\int_{\rho_0}^{\rho/2}(1+o(1))\ \frac{d\lambda}{\lambda\theta(\lambda)}} \leq \frac{\int_{\rho_0}^{\rho/2}(1+o(1))\ \frac{\theta(\lambda)}{\lambda}d\lambda}{\left(\log\frac{\rho}{2\rho_0}\right)^2} \tag{4.4}$$

Inserting inequality (4.4) into (4.3) gives

$$\frac{1}{\alpha+\varepsilon} \leq \frac{2\log\rho}{\left(\log\frac{\rho}{2\rho_0}\right)^2}\int_{\rho_0}^{\rho/2}(1+o(1))\ \frac{\theta(\lambda)}{\lambda}d\lambda$$

This is the desired result.

Proof of Theorem 1.5.

Suppose we have admissible pairs $(u_1, D_1)\ldots(u_N, D_N)$ with the D_i disjoint and u_i of one sign. Let α_i and β_i be the corresponding order and asymptotic angle of the pair (u_i, D_i) respectively. Then from Theorem 4.1,

$$\sum\frac{1}{\alpha_i} \leq \frac{1}{\pi}\sum\beta_i \leq 2$$

since the D_i are disjoint. Hence if $\alpha_1 < 1$ there is no other disjoint solution pair while if $\alpha_1 = 1$ there are at most two such pairs and necessarily $\alpha_2 = 1$, $\beta_1 = \beta_2 = \pi$.

5. Sketch of the proof of Theorem 1.9

In this section we will briefly indicate the modifications necessary to prove Theorem 1.9. For simplicity , suppose $u = 0$ on $\partial D \cap \{|z| > 1\}$ and set

$$\tilde{D_\rho} = D \cap \{1 < |z| < \rho\} \tag{5.1}$$

$$\tilde{E} = \int_{\tilde{D}_\rho}\frac{|\nabla u|^2}{W}\ dx + \int_{C_1}\frac{uu_r}{W}\ d\theta = \int_{C_\rho}\rho\frac{uu_r}{W}\ d\theta \tag{5.2}$$

$$\tilde{U} = \frac{\tilde{E}}{I} \tag{5.3}$$

Then Lemmas 2.1 and Lemma 3.2 remain valid for I , $\tilde{E}$, $\tilde{U}$. The first change comes in Lemma 3.3 . Now we have the two possible conclusions, $\tilde{U} \geq \frac{1}{2} + o(1)$ as before, or $\tilde{U} \leq -\frac{1}{2} + o(1)$. In the former case, things proceed as before without change. In the latter case, we conclude from $\rho\frac{I'}{I} = 2\tilde{U} + o(1)$ that

$$I \leq I(\rho_0)\ (\frac{\rho_0}{\rho})^{1+o(1)}$$

In other words, I decays to zero like $\frac{1}{\rho}$ contradicting our assumption.

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